Putnam Problem-Solving Seminar
Week 3
Modular Arithmetic (continued)

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1. Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.
2. Let $T_n = 2^n + 1$ for all positive integers. Let φ be the Euler φ-function, and let k be any positive integer. Let $m = n + k\varphi(T_n)$. Show that T_m is divisible by T_n.
3. If a, b, c, d are positive integers, show that 30 divides $a^{4b+d} - a^{4c+d}$.
4. Find the smallest integer n such that $2^n - 1$ is divisible by 47.
5. Let n be a positive integer such that $n + 1$ is divisible by 24. Prove that the sum of the divisors of n is divisible by 24.
6. Let $f(n)$ be the sum of the first n terms of the sequence $0, 1, 1, 2, 2, 3, 3, \ldots$, where the n-th term is given by $a_n = n/2$ if n is even and $a_n = (n-1)/2$ if n is odd. Show that if x and y are positive integers, then $xy = f(x+y) - f(x-y)$.
7. Find all solutions of $x^{n+1} - (x+1)^n = 2001$ in positive integers x and n.
8. Prove that 7, 19, and 1295 can not be written as the sum of two squares.
9. A triangular number is an integer of the form $n(n+1)/2$. Show that m is a sum of two triangular numbers if and only if $4m + 1$ is a sum of two squares.