Putnam Problem-Solving Seminar Week 2
The Euclidean Division Algorithm and an Introduction to Modular Arithmetic

There are too many problems to consider each one in one session alone. Just pick a few problems that you like and try to solve them. However, you are not allowed to work on a problem that you already know how to solve.

Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra.

Problems.

1. If $gcd(a, b) = 1$, prove that:
   (a) $gcd(a - b, a + b) \leq 2$.
   (b) $gcd(a^2 - ab + b^2, a + b) \leq 3$.
2. Let S be a nonempty set of integers such that
   (i) the difference $x - y$ is in S whenever x and y are in S, and
   (ii) all multiples of x are in S whenever x is in S.

   Prove that there is an integer d in S such that S consists of all multiples of d.
3. Consider the set $S = \{ma + nb \mid m$ and n are positive integers $\}$. Show that the previous statement (see previous problem) applies to this set and that the resulting d is $gcd(a, b)$.
4. Prove that any two successive Fibonacci numbers F_n, F_{n+1} are relatively prime. Recall that the Fibonacci numbers are given by the recursion relation $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

5. Prove that $\frac{a+b}{c+d}$ is irreducible if $ad - bc = 1$.

6. Find the smallest positive integer a for which

$$1001x + 770y = 1,000,000 + a$$

is possible, and show that it has then 100 solutions in positive integers.

7. Prove that any subset of 55 numbers chosen from the set $\{1, 2, 3, \ldots, 100\}$ must contain two numbers differing by 9.

8. Show that some positive multiple of 21 has 241 as its final three digits.

9. Prove that for any set of n integers, there is a subset of them whose sum is divisible by n.

10. Prove that if $2n + 1$ and $3n + 1$ are both perfect squares, then n is divisible by 40.

11. Suppose that the measure of a given angle is $\frac{180°}{n}$, where n is a positive integer not divisible by 3. Prove that the angle can be trisected by Euclidean means (straightedge and compass).

12. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y.