

## Putnam Problem-Solving Seminar Week 1 Induction and the Pigeon-Hole Principle

There are too many problems to consider each one in one session alone. Just pick a few problems that you like and try to solve them. However, you are not allowed to work on a problem that you already know how to solve.

**Hints.** Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra.

### Problems.

1. Choose any eight positive integers from 1 to 100. Prove that among your eight numbers, it is possible to choose two of them whose ratio lies between 1 and 2.
2. Choose any fifty-one positive integers from 1 to 100. Prove that among your fifty-one numbers, it is possible to choose two of them such that one is an integer multiple of the other.
3. Consider the sequence defined by  $a_1 = 1$  and  $a_n = \sqrt{2a_{n-1}}$ . Prove that  $a_n < 2$  for all  $n$ .
4. Prove that in a room with  $n$  people, at least two know exactly the same number of people. Assume that knowing is a mutual relation: If A knows B, then B knows A.
5. The Euclidean plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to color each of these regions either red or blue in such a way that no two adjacent regions have the same color.
6. Prove that the equation  $x^2 + y^2 = z^n$  has a solution in positive integers  $(x, y, z)$  for all  $n = 1, 2, 3, \dots$

7. Prove that  $2!4!6! \cdots (2n)! \geq ((n+1)!)^n$ .
8. How many breaks does it take to divide an  $m \times n$  chocolate bar into  $1 \times 1$  chunks?
9. Prove that  $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$  is a integer for  $n = 0, 1, \dots$
10. Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.
11. Consider any five points  $P_1, P_2, P_3, P_4, P_5$  in the interior of a square  $S$  of side length 1. Denote by  $d_{ij}$  the distance between the points  $P_i$  and  $P_j$ . Prove that at least one of the distances  $d_{ij}$  is less than  $\frac{\sqrt{2}}{2}$ .
12. Prove that there exist integers  $a, b, c$ , not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$