Putnam Problem-Solving Seminar
Power Series
Tuesday, October 30, 2007

There are too many problems to consider each one in one session alone. Just pick a few problems that you like and try to solve them. However, you are not allowed to work on a problem that you already know how to solve.

Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra.

Some common power series that we will need to know.

- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x
- $\frac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$
- $\sin x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x
- $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x
- $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n+1}$ for $-1 \leq x \leq 1$
- $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ for $-1 < x < 1$
- $(1+x)^r = \displaystyle\sum_{n=0}^{\infty} \binom{r}{n} x^n$ for r real, $-1 < x < 1$

Problems.

1. Prove that e is irrational.
2. Evaluate $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(3/5)^{4n}}{n!}$.
3. Evaluate $1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\dots$
4. Evaluate $\lim_{x\to\infty}[(e/2)x+x^2((1+1/x)^x-e)]$
5. Find the power series expansion for $f(x)=\frac{1}{x^2+5x+6}$.
6. Find the power series expansion for $f(x)=\frac{1+x}{(1+x^2)(1-x)^2}$.
7. Use the power series expansion for $\arctan x$ to find a series of rational numbers which converges to π.
8. Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{(-4\pi^2r^2)^n}{(2n+1)!}$, where r is a non-zero integer.
9. Evaluate $1+\frac{1}{3}+\frac{1\times 3}{2!3^2}+\frac{1\times 3\times 5}{3!3^3}+\dots$.
10. Sum the infinite series
$$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\dots$$
11. Evaluate $\frac{1^2}{0!}+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\dots$
12. Sum the finite series $a_0+a_1+\dots a_n$, where $a_0=2$, $a_1=5$, and for $n>1$, $a_n=5a_{n-1}-6a_{n-2}$. Hint: remember Professor Jones' talk on generating functions.
13. Evaluate in closed form: $S_n=\displaystyle\sum_{k=0}^{n}(-4)^k\binom{n+k}{2k}$. Hint: generating functions.
14. Use the technique of generating functions to show that the n-th Fibonacci number F_n is equal to
$$F_n=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

15. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

Hint: Consider $g(x) = e^{e^x}$.

16. Prove that

$$\left(\frac{1+x}{1-x}\right)^3 = 1 + 6x + 18x^2 + \ldots + (4n^2+2)x^n + \ldots$$

for $|x| < 1$.

17. Show that the power series representation for the infinite series

$$\sum_{n=0}^{\infty} x^n \frac{(x-1)^{2n}}{n!}$$

cannot have three consecutive zero coefficients.