Power Series as Functions, Part 2

Practice Problems

Solutions

1. (a) $f(x) = \dfrac{1}{(1+x)^2} = (1+x)^{-2}$

let $g(x) = -\dfrac{1}{1+x} = -1 \cdot (1+x)^{-1}$

Then $f(x) = g'(x)$. So we first find a power series for $g(x)$, then differentiate to obtain a power series for $f(x)$.

$$g(x) = -1 \cdot \frac{1}{1+x} = -1 \cdot \frac{1}{1-(-x)} = -1 \cdot \sum_{k=0}^{\infty} (-x)^k$$

$$= \sum_{k=0}^{\infty} -1 \cdot (-1)^k x^k = \sum_{k=0}^{\infty} (-1)^{k+1} x^k = -1 + x - x^2 + x^3 - x^4 + \cdots$$

Interval of convergence: $|-x| < 1 \rightarrow |x| < 1 \rightarrow (-1, 1)$

$\Rightarrow R = 1$

$$f(x) = g'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^{k+1} x^k \right)$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^{k+1} k x^{k-1}} \quad \text{This form is fine.}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \cdot k \cdot x^{k-1}$$

$$= \sum_{k=0}^{\infty} (-1)^{k+2} (k+1) x^k$$

$$= 1 - 2x + 3x^2 - 4x^3 + \cdots$$

By the Theorem, $R = 1$.

(b) $f(x) = \dfrac{1}{(1+x)^3} = (1+x)^{-3}$

Let $g(x) = -\frac{1}{2}(1+x)^{-2} = -\frac{1}{2} \cdot \frac{1}{(1+x)^2}$.

Then $f(x) = g'(x)$. So we first find a power series for $g(x)$, then differentiate to obtain a power series for $f(x)$.

$$g(x) = -\frac{1}{2} \cdot \frac{1}{(1+x)^2} = -\frac{1}{2} \cdot \sum_{k=0}^{\infty} (-1)^{k+2}(k+1)x^k$$

$$= \sum_{k=0}^{\infty} -\frac{1}{2} \cdot (-1)^{k+2}(k+1)x^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{2} \cdot (-1)^{k+3}(k+1)x^k$$

$$= -\frac{1}{2} + x - \frac{3}{2}x^2 + 2x^3 - \frac{5}{2}x^4 + \cdots$$

$\cancel{f(x) = g'(x) =}$

$$f(x) = g'(x) = \sum_{k=0}^{\infty} \frac{1}{2}(-1)^{k+3}(k+1) \cdot k \cdot x^{k-1}$$

$$= 1 - 3x + 6x^2 - 10x^3 + \cdots$$

By the theorem, $R = 1$.

(c) $$f(x) = \frac{x^2}{(1+x)^3} = x^2 \cdot \frac{1}{(1+x)^3} = x^2 \cdot \sum_{k=0}^{\infty} \frac{1}{2}(-1)^{k+3}(k+1) \cdot k \cdot x^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{2} \cdot (-1)^{k+3}(k+1) \cdot k \cdot x^{k+1}$$

$$= x^2 - 3x^3 + 6x^4 - 10x^5 + \cdots$$

2. $f(x) = \dfrac{x^3}{(1-2x)^2}$.

First, we'll find a power series for

$h(x) = \dfrac{1}{(1-2x)^2} = (1-2x)^{-2}$, then multiply by

x^3 to obtain a power series for $f(x)$.

Let $g(x) = \frac{1}{2} \cdot (1-2x)^{-1} = \frac{1}{2} \cdot \frac{1}{1-2x}$.

Then $h(x) = g'(x)$. So we first find a power series for $g(x)$, then differentiate to find a power series for $h(x)$.

$$g(x) = \frac{1}{2} \cdot \frac{1}{1-2x} = \frac{1}{2} \cdot \sum_{k=0}^{\infty} (2x)^k = \sum_{k=0}^{\infty} \frac{1}{2} \cdot 2^k \cdot x^k$$

$$= \sum_{k=0}^{\infty} 2^{k-1} x^k = \frac{1}{2} + x + 2x^2 + 2^2 x^3 + 2^3 x^4 + \cdots$$

Interval of convergence: $|2x| < 1 \to |x| < \frac{1}{2} \to (-\frac{1}{2}, \frac{1}{2})$

$R = \frac{1}{2}$.

$$h(x) = g'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} 2^{k-1} x^k\right)$$ this form is fine.

$$= \boxed{\sum_{k=0}^{\infty} 2^{k-1} \cdot k \cdot x^{k-1}} = \sum_{k=1}^{\infty} 2^{k-1} \cdot k \cdot x^{k-1} = \sum_{k=0}^{\infty} 2^k (k+1) x^k$$

$$= 1 + 2 \cdot 2x + 3 \cdot 2^2 x^2 + 4 \cdot 2^3 x^3 + \cdots$$

Finally, $f(x) = x^3 \cdot h(x)$, so:

$$f(x) = x^3 \cdot \sum_{k=0}^{\infty} 2^k (k+1) x^k = \sum_{k=0}^{\infty} 2^k (k+1) x^{k+3}$$

$$= x^3 + 2 \cdot 2 \cdot x^4 + 3 \cdot 2^2 \cdot x^5 + 4 \cdot 2^3 \cdot x^6 + \cdots$$

$$R = \frac{1}{2}.$$

3. $\int \frac{x}{1-x^8}\,dx$. First we'll find a power series representation for $\frac{x}{1-x^8}$.

$$\frac{x}{1-x^8} = x \cdot \frac{1}{1-x^8} = x \cdot \sum_{k=0}^{\infty} (x^8)^k = \sum_{k=0}^{\infty} x^{8k+1}$$

$$= x + x^9 + x^{65} + x^{8 \cdot 3+1} + \cdots$$

Interval of convergence: $|x^8| < 1 \rightarrow (-1, 1)$

$R = 1.$

$$\int \frac{x}{1-x^8}\,dx = \int \sum_{k=0}^{\infty} x^{8k+1}\,dx = \sum_{k=0}^{\infty} \int x^{8k+1}\,dx$$

$$\cancel{= \sum_{k=0}^{\infty} \frac{1}{8k}} = \sum_{k=0}^{\infty} \frac{1}{8k+2} x^{8k+2} + C$$

$$= C + \frac{1}{2}x^2 + \frac{1}{10}x^{10} + \frac{1}{\cancel{65}\ 18}x^{\cancel{65}\ 18} + \cdots$$

$R = 1.$