Power Series as Functions, Part 2

Starting with the power series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k, \quad |x| < 1, \tag{1}$$

we have been able to obtain numerous other power series by algebraic manipulation. We can also obtain power series for functions by differentiating and integrating known power series. The following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**. Also, the theorem says that the radius of convergence of a power series remains the same when a power series is differentiated or integrated.

Theorem. If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$, then the function

$$S(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is both differentiable and integrable, and

- $S'(x) = \displaystyle\sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, and the radius of convergence of the series $S'(x)$ is R
- $\displaystyle\int S(x)\,dx = C + \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1} = C + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots$, and the radius of convergence of the series $\int S(x)\,dx$ is R

Examples.

1. Find a power series representation for the function $f(x) = \ln(1+x)$. Find the radius of convergence of the power series.
2. Find a power series representation for the function $f(x) = \frac{1}{(1-x)^2}$. Find the radius of convergence of the power series.
3. Find a power series representation for the function $f(x) = \tan^{-1} x$. Find the radius of convergence of the power series.
4. Evaluate $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series.

Practice Problems.

1. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}.$$

   (b) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^3}.$$

   (c) Use part (b) to find a power series representation for

$$f(x) = \frac{x^2}{(1+x)^3}.$$

2. Find a power series representation for the function

$$f(x) = \frac{x^3}{(1-2x)^2}.$$

   Determine the radius of convergence of the power series.

3. Evaluate the indefinite integral $\displaystyle\int \frac{x}{1-x^8}\,dx$ using power series. What is the radius of convergence?