Power Series as Functions, Part 1

Geometric series. For any z such that $|z| < 1$,

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \cdots$$

is a convergent geometric series with

$$a = 1 \text{ and } r = z.$$

Thus, for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k. \tag{1}$$

This is called a **power series representation** for the function $f(z) = \frac{1}{1-z}$. We can use this basic result to find power series representations for other functions.

Examples.

1. Find a power series representation for the function $f(x) = \frac{1}{1-x}$. Find the interval of convergence of the power series.
2. Find a power series representation for the function $f(x) = \frac{1}{1+x}$. Find the interval of convergence of the power series.
3. Find a power series representation for the function $f(x) = \frac{1}{1+x^4}$. Find the interval of convergence of the power series.
4. Find a power series representation for the function $f(x) = \frac{1}{x+3}$. Find the interval of convergence of the power series.
5. Find a power series representation for the function $f(x) = \frac{x^3}{x+3}$. Find the interval of convergence of the power series.
6. Find a power series representation for the function $f(x) = \frac{1}{1+4x^2}$. Find the interval of convergence of the power series.

Practice Problems.

1. Find a power series representation for the function $f(x) = \frac{1}{1-x^3}$. Find the interval of convergence of the power series.
2. Find a power series representation for the function $f(x) = \frac{1}{x-5}$. Find the interval of convergence of the power series.

3. Find a power series representation for the function $f(x) = \frac{x}{9+x^2}$. Find the interval of convergence of the power series.

Answers.

1. $\displaystyle\sum_{k=0}^{\infty} x^{3k} = 1 + x^3 + x^6 + \cdots, \quad (-1, 1)$

2. $\displaystyle -\sum_{k=0}^{\infty} \frac{1}{5^{k+1}} x^k = -\frac{1}{5} - \frac{x}{25} - \frac{x^2}{5^3} - \cdots, \quad (-5, 5)$

3. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{1}{9^{k+1}} x^{2k+1} = \frac{x}{9} - \frac{x^3}{9^2} + \frac{x^5}{9^3} - \frac{x^7}{9^4} + \cdots, \quad (-3, 3)$