# Introduction to Power Series

A **power series** is a series of the form

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots,$$

where $x$ is a variable, the constants $a_k$ (which might depend on $k$) are called the **coefficients**, and the constant $x_0$ is called the **base point** or **center** of the power series. A power series may **converge** for some values of $x$ and diverge for other values of $x$.

**Examples.**

1. Find all values of $x$ for which the power series $\displaystyle\sum_{k=0}^{\infty} x^k$ converges.
2. Find all values of $x$ for which the power series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges.
3. Find all values of $x$ for which the power series $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k^2 3^k}$ converges.
4. Find all values of $x$ for which the power series $\displaystyle\sum_{k=0}^{\infty} (x-5)^k$ converges.
5. Find all values of $x$ for which the power series $\displaystyle\sum_{k=0}^{\infty} 2^k x^k$ converges.
6. Find all values of $x$ for which the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$ converges.
7. Find all values of $x$ for which the power series $\displaystyle\sum_{k=0}^{\infty} k! x^k$ converges.

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**Theorem: Convergence of Power Series.** For a given power series

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k,$$

there are only three possibilities:

1. The series converges only when $x = x_0$.
2. The series converges for all $x$.
3. There is a positive number $R$ such that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.

The number $R$ in case 3 is called the **radius of convergence** of the power series. The radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case 2. The **interval of convergence** of a power series is the interval that consists of all values of $x$ for which the series converges. In case 1, the interval consists of just the single point $x_0$. In case 2, the interval is $(-\infty, \infty)$. In case 3, there are four possibilities for the interval of convergence:

$$(x_0 - R, x_0 + R), \quad (x_0 - R, x + 0 + R], \quad [x_0 - R, x_0 + R), \quad [x_0 - R, x_0 + R].$$

**Examples.**

1. If $\sum_{k=0}^{\infty} c_k 4^k$ converges, does it follow that the following series converge?

   (a) $\sum_{k=0}^{\infty} c_k (-2)^k$

   (b) $\sum_{k=0}^{\infty} c_k (-4)^k$

2. Suppose that $\sum_{k=0}^{\infty} c_k x^k$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

   (a) $\sum_{k=0}^{\infty} c_k$

   (b) $\sum_{k=0}^{\infty} c_k 8^k$

   (c) $\sum_{k=0}^{\infty} c_k (-3)^k$

   $\sum_{k=0}^{\infty} c_k (-1)^k 9^k$