# Math 347
# The Poisson Process

Let $X$ be a discrete random variable that takes values in the set $\Omega = \{0, 1, 2, \ldots\}$. Then $X$ is a **Poisson random variable** with parameter $\lambda > 0$ if

$$\mathbb{P}(X = i) = p_i = \frac{e^{-\lambda}\lambda^i}{i!}$$

for $i = 0, 1, 2, \ldots$. We will use the Poisson random variable to construct the **Poisson process**, which has important applications in mathematical modeling and queuing theory.

A **stochastic process**

$$\{X(t) : t \in T\}$$

is a collection of random variables, i.e. for each $t \in T$, $X(t)$ is a random variable. The index $t$ is often interpreted as time, and as a result, we shall refer to $X(t)$ as the *state* of the process at time $t$. For example, $X(t)$ might be the total number of customers that have entered the market by time $t$, or the number of customers in the market at time $t$, or the total amount of stales that have been recorded in the market by time $t$.

The set $T$ is called the *index set* of the process. If $T$ is a countable set, then the stochastic process is a *discrete-time* process. If $T$ is an interval of the real numbers, then the stochastic process is a *continuous-time process*. A stochastic process can be thought of as a family (or collection) or random variables that describes the evolution through time of some process.

A stochastic process

$$\{N(t) : t \geq 0\}$$

is said to be a **counting process** if $N(t)$ satisfies the following properties:

- $N(t) \geq 0$.
- $N(t)$ is integer-valued.
- If $s < t$, then $N(s) \leq N(t)$.
- For $s < t$, $N(t) - N(s)$ is equal to the number of events that have occurred in the interval $(s, t)$.

Some examples of counting processes are the following:

- Let $N(t)$ be the number of people who have entered a particular store at or prior to time $t$. Then $\{N(t) : t \geq 0$ is a counting process in which an event corresponds to a person entering the store. Note that if we let $N(t)$ be the number of people in the store at time $t$, then $\{N(t) : t \geq 0\}$ would *not* be a counting process. Why not?

- Let $N(t)$ be the number of people who have been born by time $t$. Then $\{N(t) : t \geq 0\}$ is a counting process in which an event corresponds to a person being born. Does $N(t)$ count people that have died by time $t$? Explain why it must.
- If $N(t)$ is the number of goals that a given soccer player has scored by time $t$, then $\{N(t) : t \geq 0\}$ is a counting process in which an event corresponds to the player scoring a goal.

A counting process is said to posses **independent increments** if the number of events which occur in disjoint time intervals are independent.

One of the most important counting processes is the **Poisson process** which is defined as follows. The counting process $\{N(t) : t \geq 0\}$ is said to be a **Poisson process with rate** $\lambda$, $\lambda > 0$ if

- $N(0) = 0$.
- The process has independent increments.
- The number of events in any interval of length $t$ is a Poisson random variable with parameter $\lambda t$, i.e. for all $s, t \geq 0$,

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

  for $n = 0, 1, 2, \ldots$.

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**Problems.**

1. Find $\mathbb{E}[N(t)]$ for a Poisson process $N(t)$ with rate $\lambda$. Interpret in terms of the fact that the Poisson process is often used to model arrival processes.
2. Suppose that customers arrive at an office according to a Poisson process with rate 20 per hour. Compute the following:
   (a) The average number of arrivals between 8:00 am and 10:00 am.
   (b) The probability that no customer will arrive between 8:00 am and 8:10 am.
   (c) The probability that one customer will arrive between 8:10 am and 8:20 am given that no customers have arrived between 8:00 am and 8:10 am.
3. Suppose that $N(t)$ is a Poisson process with rate $\lambda$. Let $T_1$ denote the time of the first event. Find $\mathbb{P}[T_1 > t]$. What does this represent in terms of the process $N(t)$? Let $T_2$ denote the time of the second event. Find $\mathbb{P}[T_2 - T_1 > t]$. Hint: start by computing $\mathbb{P}[T_2 - T_1 > t | T_1 = s]$. Generalize.
4. Shipments of paper arrive at a printing shop according to a Poisson process with a mean of 0.5 shipments per day.

(a) Find the probability that the printing shop receives more than two shipments per day.

(b) If there are more than 4 days between shipments, all the paper will be used up, and the presses will be idle. Find the probability that this will happen.

5. The number of times that a piece of military hardware must be serviced in an arctic environment is a Poisson process with an average of one service every 200 hours of operation.

(a) If a mission involving the use of this hardware requires 24 hours for completion, find the probability that the mission will be completed without having to service the hardware.

(b) If the probability that no service is required during a mission is 0.95, find the length of time required for the mission.

6. Consider a Poisson process $N(t)$ with rate $\lambda$, and suppose that each time an event occurs, it is classified as either a type 1 or type 2 event. Suppose further that the probability that an event is classified as a type 1 event is $p$ and the probability that an event is classified as a type 2 event is $1 - p$. Let $X_1(t)$ and $X_2(t)$ denote the number of type 1 and type 2 events, respectively, occurring in $[0, t]$. Show that the processes $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are both Poisson processes with rates $\lambda p$ and $\lambda(1-p)$, respectively. Show that $X_1(t)$ and $X_2(t)$ are independent.

7. A hospital serves a small city. Emergencies requiring an ambulance arise according to a Poisson process with rate $\lambda = 3$ per hour. Each emergency is on the west side of the town with probability 0.6 and is on the east side with probability 0.4. The hospital has a WEST team of 2 ambulances assigned to emergencies on the west side of town, and an EAST team of 1 ambulance assigned to emergencies on the east side of town.

(a) What is the probability that there are 4 or more emergencies in the city in one hour?

(b) What is the probability that one or both teams has more emergencies than it can handle in one hour? Assume that an ambulance dispatched to one emergency cannot respond to any others for at least one hour.