Math 224
Properties of Orthogonal Matrices

Some of the following statements are true, and some are false. Some of those that are false can be modified slightly to make a true statement. By experimenting in Maple, and by using what you know about orthogonal matrices, dot products, eigenvalues, determinants, etc., verify, contradict, or improve the following statements. For those that you believe to be true, you should come up with a proof that you could present at the board. For those that you believe to be false, you should come up with a counterexample, and think about whether or not the statement could be modified slightly to produce a true statement.

1. The identity matrix is orthogonal.

2. The matrix
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
   is orthogonal.

3. The matrix
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$
   where θ is any angle, is orthogonal.

4. Every diagonal matrix is orthogonal.

5. If A is an $n \times n$ orthogonal matrix, and $\mathbf{x}$ and $\mathbf{y}$ are any column vectors in $\mathbf{R}^n$, then $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

6. If A is an $n \times n$ orthogonal matrix, and $\mathbf{x}$ is any column vector in $\mathbf{R}^n$, then $||A\mathbf{x}|| = ||\mathbf{x}||$. It may be useful to remember that the command in Maple for finding $||\mathbf{v}||$ is norm(v,2).

7. If A is an $n \times n$ orthogonal matrix, and $\mathbf{x}$ and $\mathbf{y}$ are any non-zero column vectors in $\mathbf{R}^n$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is equal to the angle between $A\mathbf{x}$ and $A\mathbf{y}$. It may be useful to remember that the command in Maple for finding the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$ is angle(v,w).

8. An orthogonal matrix must be symmetric.

9. The product of two orthogonal matrices is also orthogonal.

10. The norm of the first column of an orthogonal matrix must be 1.

11. The norm of the first row of an orthogonal matrix must be 1.

12. The vectors formed by the first and last rows of an orthogonal matrix must be orthogonal.

13. The vectors formed by the first row and the second column of an orthogonal matrix must be orthogonal.

14. The determinant of an orthogonal matrix is always 1.

15. Every entry of an orthogonal matrix must be between 0 and 1.

16. The eigenvalues of an orthogonal matrix are always ± 1.

17. If the eigenvalues of an orthogonal matrix are all real, then the eigenvalues are always ± 1.

18. In any column of an orthogonal matrix, at most one entry can be equal to 1.

19. In any column of an orthogonal matrix, at most one entry can be equal to 0.

20. If A is an $n \times n$ symmetric orthogonal matrix, then $A^2 = I$.

21. If A is an $n \times n$ symmetric matrix such that $A^2 = I$, then A is orthogonal.

22. The nullspace of any orthogonal matrix is $\{\mathbf{0}\}$.

23. If A is a 2×2 orthogonal matrix with determinant 1, then A is an orthogonal matrix.