Math 333
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Introduction to Non-homogeneous Second Order Linear Equations

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Note: The material presented in these lecture notes corresponds (approximately) to Section 4.1 (Forced Harmonic Oscillators) and Section 4.2 (Sinusoidal Forcing) in your textbook. Section 4.2 presents an alternative method for solving second-order linear non-homogeneous differential equations when the forcing function $g(t)$ is a sine or cosine function. We will discuss the method of undetermined coefficients in class, but you are welcome to read Section 4.2 for an alternative method. You will not be responsible (on quizzes, homeworks, or tests) for the complexification method presented in Section 4.2, though you are welcome to use the complexification method if you prefer it over the undetermined coefficients method.

Consider the general **second-order linear** differential equation:

$$y'' + p(t)y' + q(t)y = g(t), \tag{1}$$

where $p(t)$, $q(t)$, and $g(t)$ are given continuous functions on some open interval I.

We will often refer to the differential equation in Eqn. (1) as a **forced** differential equation, and we will refer to $g(t)$ as the **forcing function**. Later, we will discuss physical and qualitative results regarding second-order linear forced differential equations with constant coefficients. To find the general solution of the differential equation in Eqn. (1), we first find the general solution of the **associated homogeneous** second-order linear differential equation:

$$y'' + p(t)y' + q(t)y = 0. \tag{2}$$

The Extended Linearity Principle. Suppose that

$$k_1y_1(t) + k_2y_2(t)$$

is the general solution of the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Further, suppose that $y_p(t)$ is a **particular solution** of the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Then the general solution of the original non-homogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is

$$y(t) = k_1 y_1(t) + k_2 y_2(t) + y_p(t).$$

Proof. First, show that $y(t) = k_1 y_1(t) + k_2 y_2(t) + y_p(t)$ is indeed a solution of the non-homogeneous differential equation $y'' + p(t)y' + q(t)y = g(t)$. Since $y(t)$ contains two arbitrary constants, it is the general solution.

Summary. To find the general solution of an arbitrary second-order non-homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

use the following procedure:

1. Find the general solution
$$k_1 y_1(t) + k_2 y_2(t)$$
of the associated homogeneous equation
$$y'' + p(t)y' + q(t)y = 0.$$
2. Find *some* particular solution $y_p(t)$ of the non-homogeneous equation.
3. The general solution is then
$$y(t) = k_1 y_1(t) + k_2 y_2(t) + y_p(t).$$

Example 1. Find the general solution of the differential equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution. First, find the general solution of the associated homogeneous equation

$$y'' - 3y' - 4y = 0.$$

The general solution is

$$k_1 e^{4t} + k_2 e^{-t}.$$

Next, find a particular solution $y_p(t)$ of the original non-homogeneous differential equation. To do this, guess a solution of the form

$$y_p(t) = \alpha e^{2t},$$

and substitute into the original differential equation to determine what the coefficient α must be equal to. We obtain $\alpha = -1/2$, so a particular solution is

$$y_p(t) = -\frac{1}{2} e^{2t}.$$

Thus, the general solution of the given differential equation is

$$y(t) = k_1 e^{4t} + k_2 e^{-t} - \frac{1}{2} e^{2t},$$

where k_1 and k_2 are arbitrary constants.

Note: The method used in Example 1 to find a particular solution $y_p(t)$ of the non-homogeneous equation is called the **method of undetermined coefficients**. We make an initial assumption about the *form* of the particular solution $y_p(t)$, but leave the coefficients unspecified. We then substitute the assumed expression into Eqn. (1), and try to determine the coefficients so as to satisfy the differential equation. If we are successful, then we have found a particular solution of the differential equation. If we cannot determine the coefficients, then there is no solution of the form that we initially assumed. In this case, we may modify the initial assumption and try again, as the next example illustrates.

Example 2. Find the general solution of the differential equation

$$y'' - 3y' - 4y = 2e^{-t}.$$

Solution. As in Example 1, the general solution of the associated homogeneous equation is

$$k_1 e^{4t} + k_2 e^{-t}.$$

To find a particular solution of the non-homogeneous equation, we guess a solution of the form

$$y_p(t) = \alpha e^{-t}.$$

Substituting this expression in the differential equation, we obtain

$$0e^{-t} = 2e^{-t}.$$

There is no choice of α that satisfies this equation, so there is no particular solution of the assumed form. For reasons that will become clear soon, we modify our assumed particular solution by multiplying it by t; that is, we assume a particular solution of the form

$$y_p(t) = \alpha t e^{-t}.$$

Substituting this expression in the differential equation, we obtain $\alpha = -2/5$. Thus,

$$y_p(t) = -\frac{2}{5} t e^{-t}$$

is a particular solution of the non-homogeneous differential equation. The general solution is

$$y(t) = k_1 e^{4t} + k_2 e^{-t} - \frac{2}{5} t e^{-t}.$$

In this example, why was it not possible to solve for α so that $y_p(t) = \alpha e^{-t}$ is a solution of the non-homogeneous equation? The assumed particular solution $y_p(t) = \alpha e^{-t}$ is actually a solution of the associated homogeneous equation, so it cannot possibly be a solution of the non-homogeneous equation.

Note: The outcome of the previous example suggests the following. If the assumed form of the particular solution duplicates a solution of the corresponding homogeneous equation, modify the assumed particular solution by multiplying it by t. Occasionally, this modification will be insufficient to remove all duplication with the solutions of the homogeneous equation, in which case it is necessary to multiply by t a second time. For a second order equation, it will never be necessary to carry the process further than this.

Example 3. Find the general solution of the differential equation

$$y'' - 3y' - 4y = 2\sin t.$$

Solution. As in Examples 1 and 2, the general solution of the associated homogeneous equation is

$$k_1 e^{4t} + k_2 e^{-t}.$$

To find a particular solution of the non-homogeneous equation, we guess a solution of the form

$$y_p(t) = \alpha \sin t + \beta \cos t.$$

Substituting this expression into the differential equation, we obtain $\alpha = -5/17$ and $\beta = 3/17$. Thus,

$$y_p(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$$

is a particular solution of the non-homogeneous differential equation. The general solution is

$$y(t) = k_1 e^{4t} + k_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Guidelines for choosing the form of the particular solution $y_p(t)$.

- If $g(t)$ is an exponential function e^{at}, assume that $y_p(t)$ is of the form αe^{at}.
- If $g(t)$ is a sine or cosine function, $\sin(at)$ or $\cos(at)$, assume that $y_p(t)$ is of the form $\alpha \sin(at) + \beta \cos(at)$.
- If $g(t)$ is a polynomial function, assume that $y_p(t)$ is a polynomial of the same degree.
- If the initial assumed form duplicates a solution of the corresponding homogeneous equation, modify the assumed particular solution by multiplying by t. If necessary, multiply by t a second time.

- If $g(t)$ is a product of an exponential, sine or cosine, or polynomial function, assume that $y_p(t)$ is a product of the corresponding forms described above.

Example 4. Find the general solution of the differential equation

$$y'' - 3y' - 4y = 3t + 2.$$

Solution. As in Examples 1-3, the general solution of the associated homogeneous equation is

$$k_1 e^{4t} + k_2 e^{-t}.$$

To find a particular solution of the non-homogeneous equation, we guess a solution of the form

$$y_p(t) = \alpha t + \beta.$$

Substituting this expression into the differential equation, we obtain $\alpha = -3/4$ and $\beta = 1/16$. Thus

$$y_p(t) = -\frac{3}{4}t + \frac{1}{16}$$

is a particular solution. The general solution of the differential equation is

$$y(t) = k_1 e^{4t} + k_2 e^{-t} - \frac{3}{4}t + \frac{1}{16}.$$

Example 5. Find the general solution of the differential equation

$$y'' - 3y' - 4y = -8e^t \cos(2t).$$

Solution. As in Examples 1-4, the general solution of the associated homogeneous equation is

$$k_1 e^{4t} + k_2 e^{-t}.$$

To find a particular solution of the non-homogeneous equation, we guess a solution of the form

$$y_p(t) = \alpha e^t \cos(2t) + \beta e^t \sin(2t).$$

Substituting this expression into the differential equation, we obtain $\alpha = 10/13$ and $\beta = 2/13$. Thus,

$$y_p(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t)$$

is a particular solution of the non-homogeneous differential equation. The general solution is

$$y(t) = k_1 e^{4t} + k_2 e^{-t} + \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t).$$

(**4.1 #36:**) Next, we consider the case in which the forcing function $g(t)$ is a *sum* of two terms:

$$g(t) = g_1(t) + g_2(t).$$

More precisely, suppose that $y_1(t)$ is a solution of the differential equation

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and that $y_2(t)$ is a solution of the differential equation

$$y'' + p(t)y' + q(t)y = g_2(t).$$

Show that $y_1(t) + y_2(t)$ is a solution of the differential equation

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t).$$

Note that a similar conclusion holds if $g(t)$ is the sum of any finite number of terms.

Example 6. Use the technique described above to find the general solution of the differential equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos(2t).$$

Solution. As in Examples 1-5, the general solution of the associated homogeneous equation is

$$k_1e^{4t} + k_2e^{-t}.$$

We can write the forcing function

$$g(t) = 3e^{2t} + 2\sin t - 8e^t\cos(2t)$$

as a sum of three functions

$$g_1(t) = 3e^{2t}, \quad g_2(t) = 2\sin t, \quad g_3(t) = -8e^t\cos(2t).$$

Solutions of the differential equations

$$\begin{aligned} y'' - 3y' - 4y &= 3e^{2t} \\ y'' - 3y' - 4y &= 2\sin t \\ y'' - 3y' - 4y &= -8e^t\cos(2t) \end{aligned}$$

have been found in Examples 1, 3, and 5, respectively. Thus, a particular solution of the differential equation is the sum of the particular solutions of the differential equations above, namely

$$y_p(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t).$$

Finally, the general solution is

$$y(t) = k_1e^{4t} + k_2e^{-t} - \frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t).$$