Math 333
January 29, 2008
Linear Equations: A First Approach

Definition. A first-order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t),$$

where $a(t)$ and $b(t)$ are any functions of t.

Example 1.

Example 2.

Definition. A first-order linear DE is **homogeneous** if $b(t) = 0$ for all t, i.e. if the DE can be written in the form

$$\frac{dy}{dt} = a(t)y,$$

where $a(t)$ is any functions of t. Otherwise, the DE is **nonhomogeneous**.

Example 1.

Example 2.

The Extended Linearity Principle. Consider the nonhomogeneous first-order linear DE

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

0. First, note that the general solution of the associated homogeneous equation is

$$y_(t) = ke^{\int a(t)\,dt},$$

where k is an arbitrary constant.

1. If $y_h(t)$ is *any* solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_q(t) - y_p(t)$ is a solution of the homogeneous equation.

Conclusion: Combining these statements, the Extended Linearity Principle says that **the general solution of the nonhomogeneous first-order linear DE**

$$\frac{dy}{dt} = a(t)y + b(t)$$

is

$$ky_h(t) + y_p(t),$$

where $ky_h(t)$ is the general solution of the associated homogeneous equation, and $y_p(t)$ is one solution of the nonhomogeneous equation.

Proof. We'll prove statements (0) and (1) in class. For homework, you'll be asked to prove (2). You should also think about why the conclusion follows from the three statements.

We can use the Extended Linearity Principle to solve linear differential equations. We proceed in the following way.

1. Find the general solution $ky_h(t)$ of the associated homogeneous equation.

2. Find a particular solution $y_p(t)$ of the nonhomogeneous equation. This is typically the difficult step, and we often have to use guesswork to find $y_p(t)$.

3. The general solution of the linear differential equation is then $ky_h(t) + y_p(t)$.

Example. Find the general solution of the differential equation

$$\frac{dy}{dt} = y + 3e^{-2t}.$$