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LARGE-AMPLITUDE PERIODIC OSCILLATIONS IN SUSPENSION BRIDGES: SOME NEW CONNECTIONS WITH NONLINEAR ANALYSIS*

A. C. LAZER† AND P. J. MCKENNA‡

Abstract. This paper surveys an area of nonlinear functional analysis and its applications. The main application is to the existence and multiplicity of periodic solutions of a possible mathematical models of nonlinearly supported bending beams, and their possible application to nonlinear behavior as observed in large-amplitude flexings in suspension bridges. A second area, periodic flexings in a floating beam, also nonlinearly supported, is covered at the end of the paper.

Key words. nonlinear periodic oscillation, bending beams, multiple solutions

AMS(MOS) subject classifications. 35B10; secondary 70K30, 73K05

1. Periodic oscillation in suspension bridges: Facts old and new. If the science of mechanics has a classic movie, it must be the old film of the collapse of the Tacoma Narrows suspension bridge. Most readers have surely seen the dramatic large-scale oscillations, followed by the collapse of the structure. Recent research uncovered a compelling explanation of this phenomenon, which challenged the commonly accepted one.

There is a standard explanation of the large oscillations of the bridge. The claim is that the bridge behaves like a particle of mass one at the end of a spring with spring constant k, which is subject to a forcing term of frequency $\mu/2\pi$. This is a sophomore level problem, and we can all answer it. If μ is very close to the square root of k, then large oscillation results. If μ is not, then it does not.

The usual explanation [10] then says that the forcing term came from a train of alternating vortices being shed by the bridge as the wind blew past it. The frequency just happened to be at a value very close to a resonant frequency of the bridge. Thus, even though the magnitude of the forcing term was small, the phenomenon of linear resonance was enough to explain the large oscillation and eventual collapse of the bridge.

This explanation has enormous appeal in the mathematical and scientific community. It is plausible, remarkably easy to understand, and makes a nice example in a differential equations class. It also explains something otherwise difficult to understand. An early convert was the *New York Times.*[1]

Nonetheless, it leaves some nagging doubts. Usually, the phenomenon of linear resonance is very precise. (For example, audio tape companies advertise the accuracy of their product by showing how it reproduces the frequency with enough precision to recreate resonance.) Could it really be that such precise conditions existed in the middle of the Tacoma Narrows, in an extremely powerful storm?

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† University of Miami, Coral Gables, Florida 33124.
‡ University of Connecticut, Storrs, Connecticut 06268.

[1] Two days after the first reports, the editorial page contained the following analysis: "Like all suspension bridges, that at Tacoma both heaved and swayed with a high wind. It takes only a tap to start a pendulum swinging. Time successive taps correctly and soon the pendulum swings with its maximum amplitude. So with the bridge. What physicists call resonance was established ...." (The authors wonder what was meant by the maximum amplitude of the pendulum.)

We found more details in [44], where the classic explanation is attributed to von Kármán. "A national commission investigating the collapse included Theodore von Kármán of Caltech. He explained that vortices were pouring off the top and bottom of the bridge, driving the bridge at its resonant frequency, which eventually led to its collapse." Von Kármán did indeed say as much in his popular autobiography[2] [25]. "As I had suspected: the villain was the Kármán vortices."

The commission in question included von Kármán, and Othmar H. Amann, the architect who designed the George Washington Bridge, among many others, and Glenn B. Woodruff. They studied all the data, and reported on their conclusions to the administrator of the Federal Works Agency, John M. Carmody. Still suspicious of simple resonance, we studied their report [5].

The report was full of all sorts of data that had been painstakingly collected over the months of the bridge's existence prior to its collapse. Amplitudes, frequencies, and modes of oscillation, along with weather conditions, wind velocities and directions had all been recorded. The conclusions included the following remarkable paragraph:

> It is very improbable that resonance with alternating vortices plays an important role in the oscillations of suspension bridges. First, it was found that there is no sharp corellation between wind velocity and oscillation frequency, as is required in the case of resonance with vortices whose frequency depends on the wind velocity.... It seems that it is more correct to say that the vortex formation and frequency is determined by the oscillation of the structure than that the oscillatory motion is induced by the vortex formation.

In [52], one suspension bridge engineer comments:

> Unfortunately for the record, some of the writings of von Kármán leave a trail of confusion on this point. Perhaps the most glaring inaccuracy is his apparent insistence in his popular biography that "the culprit was the Kármán vortex street." ... However, since a body with changing angle of attack does indeed shed *motion induced* wake vortices, there was indeed a "non-Kármán" trail of vortices.

Later in the same paper, the author remarks on the difficulty of analyzing the oscillation of suspension bridges such as the Golden Gate Bridge, which exhibit self-excitation under gusting. In particular, he suggests that "The analytics [sic] of the buffeting problem are accompanied by a number of problems, *notably the possible inadequacy of the linear superposition ideas most commonly used.* (emphasis added)"

There seems to be a need to give a clear mathematical argument as to why suspension bridges oscillate. As made clear in [5], [7], suspension bridges have a history of large-scale oscillation and catastrophic failure under high and even moderate winds, as well as (less common) under other mechanical forces.

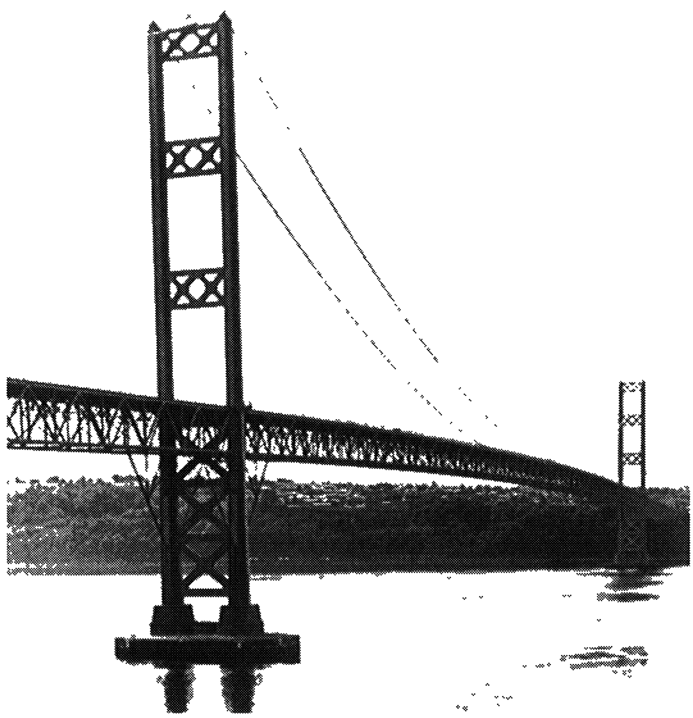
Earlier bridges such as the Bronx–Whitestone bridge, on which a traveller might often get seasick due to the large-scale motions, or the Golden Gate Bridge, which has exhibited travelling waves [5], had exhibited oscillatory behavior due to the action of wind.

What distinguished the Tacoma Narrows was the extreme flexibility of its roadbed,

---

[2] This is a highly entertaining account of one man's progress through the scientific-industrial complex. It also includes photos of von Kármán with popes, presidents, and Jayne Mansfield. Characteristically, von Kármán is looking, not at Jayne, but at the camera. The book also includes his opinions on men of science in the twentieth century. "Einstein was the greatest ... he had four great ideas. Most other great names had one, or at most, two. I had ... three and a half. (p. 4)"

(a)



(b)

FIG. 1. *The Tacoma Narrows Bridge.* (a) *The original bridge which was light, flexible, two lane, and cost* \$6 *million. Photo* ©1940 Seattle Times. (b) *The replacement bridge, which is heavy, rigid, four-lane, and cost* \$15 *million. Photo* ©1974 Seattle Times.

being an order of magnitude higher than that of earlier bridges mentioned [5]. This resulted in a pronounced tendency to oscillate vertically, under widely differing wind conditions. The bridge might be quiet in winds of forty miles per hour, and might oscillate with large amplitude in winds as low as three or four miles per hour. These vertical oscillations were standing waves of different nodal types. The report [5] contains rich detail on this type of wave. Curiously, the engineering literature refers to this type of oscillation as "benign" [52].

The second type of oscillation was observed just prior to the collapse of the bridge. This was a pronounced torsional mode. This type of oscillation was observed after the bridge went into large vertical motion which apparently induced a slippage of a crucial part of the bridge called the cable band, which attached the center of the cable to the roadbed. Under the influence of the large amplitude vertical motions (of about five feet in amplitude with a frequency of 38 per minute), this band slipped, and "the change from the moderate parallel motions of the cables to the more violent out-of-phase motions was sudden" [5, p. 58].

It should be emphasized that in the observed torsional motion, some of the cables were alternately loosening and tightening. This is the nonlinear effect that we are interested in studying.

A wind-tunnel study of a scale model of the Tacoma Narrows Bridge was studied in a variety of wind conditions by Dunn [5, Appendix VIII]. Although he was able to induce vertical motions at about the right wind velocity, he was only able to induce torsional motion in the model by making it fifty per cent more flexible, and increasing the velocity to approximately twice that of the actual storm, on the day of the failure.

There is a curious fact which we should bear in mind when attempting to model large amplitude oscillation in suspension bridges, namely that for small to medium amplitude oscillations, the behavior is almost perfectly linear [53].

Thus, there is a need for a mathematical explanation of

1. What in the nature of *suspension bridges* makes them so prone to large-scale oscillation;
2. The fact that the bridge would go into large oscillation under the impulse of a single gust, and at other times would remain motionless in winds of thirty to forty m.p.h.;
3. The fact that the motion would change rapidly from one nodal type to another;
4. The fact that large vertical oscillation could rapidly change to torsional;
5. The existence of the travelling waves;
6. The fact that the motion is linear over small to medium range oscillation.

Providing such explanations is the goal of this paper.

What distinguishes suspension bridges, we claim, is their fundamental nonlinearity. The restoring force due to a cable is such that it strongly resists expansion, but does not resist compression. Thus, the simplest function to model the restoring force of the stays in the bridge would be a constant times u, the expansion, if u is positive, but zero, if u is negative, corresponding to compression.

The stays would be in a state of tension under the weight of the bridge, rather than at u equal zero.

One area of nonlinear analysis has recently made considerable progress on problems with this type of nonlinearity. This type of nonlinearity, often called *asymmetric* (because it behaves differently for u positive and u negative), has given rise to the following quasi principle:

> Systems with asymmetry and large uni-directional loading tend to have multiple oscillatory solutions: the greater the asymmetry, the larger the number of oscillatory solutions, the greater the loading, the larger the amplitude of the oscillations.

In §2 of this survey, we shall review some standard literature in the field of semilinear partial differential equations and then describe some of these theorems in their original context. In §3, we shall describe some results of this type for simple mechanical systems.

In §4, we shall return to the bridge, describe a new differential equation which models a bridge, show how this theory applies in accordance with our stated goals, and show how this area gives key new insights into the oscillation of suspension bridges, even suggesting new ways of constructing extremely light flexible bridges which would not be prone to large-scale oscillation.

In §5, we show how similar results apply in the area of naval architecture.

Any proofs that are given in the course of this paper should be accessible to the average graduate student with some background in differential equations and functional analysis.

Throughout the paper are scattered what to the best of our knowledge are open problems. We will be happy to respond to queries on the state of knowledge of these problems in the future.

2. A review of the literature of nonlinear elliptic boundary value problems: Classical and recent. In this section, we shall review two bodies of work: the older literature on the existence of solutions to semilinear elliptic boundary value problems, and the work referred to in the introduction on systems in which an asymmetric nonlinearity can give rise to multiple solutions. The older body of work, which we shall study first falls under the category of what we call the *nearly linear* case.

2.1. Nearly linear theory. We begin with a review of literature up to the end of the 1970s on the existence of solutions for the equation

$$
\begin{aligned}
&\Delta u + f(u) = h(x) \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{1}
$$

As always throughout this article, the region Ω is a smooth, simply connected, bounded region in R^n, and the function f is assumed to be asymptotically homogeneous with limits at plus infinity and minus infinity, that is, $f'(+\infty)$ exists and is equal to b and $f'(-\infty)$ exists and is equal to a.

The Laplacian, with Dirichlet boundary conditions has eigenvalues, $\lambda_i, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and their corresponding eigenfunctions will be denoted by ϕ_i. (Recall that the first eigenvalue is simple and that the first eigenfunction, ϕ_1, is strictly positive in Ω.)

A good guide for this sort of problem is the piecewise linear equation,

$$\Delta u + bu^+ - au^- = s\phi_1. \tag{2}$$

Recall that the nonlinear function u^+ denotes the function which is u, if u is positive, and zero if u is negative and $u^- = (-u)^+$. The real number s is a parameter which we will vary.

Note that if $a, b < \lambda_1$, we can write down the solution to (2) explicitly. If $s > 0$ then the solution is $s\phi_1/(a - \lambda_1)$, and if $s < 0$, the solution is $s\phi_1/(b - \lambda_1)$. Thus, the semilinear equation admits a solution for all values of s.

This is true for all right-hand sides, not just $s\phi_1$. Following earlier work of Picard (*en route* to his celebrated method of successive approximations), Hammerstein [22] proved the following theorem.

THEOREM 2.1. *If* $\sup \mid \partial f/\partial u \mid\leq m < \lambda_1$, *then* (1) *admits a unique solution for any choice of* $h(x)$. *Moreover, if* $f'(+\infty), f'(-\infty) < \lambda_1$, *there always exists a solution, which may not be unique.*

In the interests of exposition, we have omitted some of the technical smoothness assumptions required by Hammerstein.

We could now look at the situation in (2), where a and b are no longer below λ_1. We know that if we take $a = b = \lambda_i$, then the Fredholm alternative applies, and there will be no solution if $s \neq 0$. In addition, we know that if $\lambda_i < a, b < \lambda_{i+1}$, then we can again explicitly write down the solution, namely $s\phi_1/(b-\lambda_1)$ if s is positive, and $s\phi_1/(a-\lambda_1)$ if s is negative.

This, it turns out, is a good guide for the nonlinearity $f(u)$, as was proved, almost twenty years after Hammerstein, by Dolph [19].

THEOREM 2.2. *If there exists* $\varepsilon > 0$ *such that for all s,* $\lambda_n + \varepsilon < f'(s) < \lambda_{n+1}$, *then* (1) *has a unique solution, for any choice of right-hand side. If we only know that* $\lambda_n < f'(+\infty), f'(-\infty) < \lambda_{n+1}$, *then there exists a solution which may not be unique.*

Again, some technical hypotheses are omitted. We include a sketch of the proof, since it is an elegant application of the contraction fixed point theorem.

Proof. Write (1) as $L(u) = N(u)$, where $L = -\Delta - \gamma I, \gamma = (\lambda_n + \lambda_{n+1})/2$, and $N(u) = f(u) - \gamma u - h(x)$.

Observe that in $L^2(\Omega)$, there exists an $\varepsilon > 0$ such that

$$\|N(u) - N(v)\| \leq \frac{(\lambda_{n+1} - \lambda_n - 2\varepsilon)}{2}\|u - v\|$$

and that, since L is self-adjoint, with eigenvalues $\mu_i = \lambda_i - \gamma$, satisfying

$$\mu_1 < \mu_2 \leq \cdots \mu_n = \frac{\lambda_n - \lambda_{n+1}}{2} < 0 < -\frac{\lambda_n - \lambda_{n+1}}{2} = \mu_{n+1},$$

therefore, it follows that

$$\|L^{-1}\| = \frac{2}{\lambda_{n+1} - \lambda_n}.$$

Thus, we conclude that the map $u \to L^{-1}N(u)$ is a contraction on $L^2(\Omega)$, and this proves the uniqueness part of the theorem. The second part of the theorem is proved by the Schauder fixed point theorem, using the same basic idea, namely, by showing that the same map maps a large ball into itself.

In the case in which the nonlinearity does not cross the spectrum of the Laplacian (in the sense that closure(rangef') $\cap \{\lambda_i\} = \emptyset$, the semilinear equation (1) behaved like the piecewise linear one, (2).

The next natural object of study was the case where the interval $(f'(-\infty), f'(+\infty))$ contained some of the spectrum of the Laplacian. Again, let us look to the piecewise linear case.

(3) $$\Delta u + bu^+ - au^- = s\phi_1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We observe that if $a < \lambda_1 < b$, then if $s > 0$, there exist two "linear" solutions. These are $s\phi_1/(b-\lambda_1)$ and $s\phi_1/(a-\lambda_1)$. The first of these is positive, and the second is negative, so we can verify that they are solutions by substituting them directly into the equation.

Almost as obvious is that if $s < 0$, there can be no solution. This can be seen by rewriting (2) as

$$(\Delta + \lambda_1)u + (b - \lambda_1)u^+ - (a - \lambda_1)u^- = s\phi_1.$$

Now multiply across by ϕ_1 and integrate by parts. This gives

$$0 = \int((b - \lambda_1)u^+ - (a - \lambda_1)u^-)\phi_1 dx - s,$$

since $((\Delta + \lambda_1)u, \phi_1) = (u, (\Delta + \lambda_1)\phi_1) = 0$, which is clearly impossible if $s < 0$, since all terms in the integral are positive. This also shows that if $s = 0, u = 0$ is the only solution.

The next question is, are these the only solutions to the piecewise linear problem, or could there be others?

LEMMA 2.3. *Suppose $a < \lambda_1 < b < \lambda_2$. Then if $s > 0$, there are exactly two solutions to* (2).

Proof (sketch). Let P be the orthogonal projection onto the subspace of $H = L^2(\Omega)$ spanned by ϕ_1. Then, $I - P$ is projection onto the orthogonal complement. Then, (2) is equivalent to

$$(I - P)\Delta u + (I - P)(bu^+ - au^-) = 0, \tag{4}$$

$$P((b - \lambda_1)u^+ - (a - \lambda_1)u^-) = s\phi_1. \tag{5}$$

To verify this claim, take the P and $(I - P)$ projections of (2) and observe that $P(\Delta + \lambda_1)u = 0$. Now we write $u = v + w$, where $v = Pu$, and $w = (I - P)u$. Then we obtain, from (4),

$$\Delta w + (I - P)(b(v + w)^+ - a(v + w)^-) = 0.$$

Now, regard this as an equation, for fixed v, on $(I - P)H$. Let $\gamma = (a + b)/2$, and observe, by the hypothesis of the lemma, that $\gamma < \lambda_2$. Again, we can rewrite (4) as

$$(-\Delta - \gamma)w = (I - P)((b - \gamma)(v + w)^+ - (a - \gamma)(v + w)^-).$$

It is easy to check that if L is $(-\Delta - \gamma)$, restricted to $(I - P)H$, then the norm of L^{-1} is $(\lambda_2 - \gamma)^{-1}$. Also, note that if the right-hand side of the above equation is regarded as a nonlinear map from $(I - P)H$ to itself, then it has a Lipschitz constant $(b - a)/2$. Since the product $(\lambda_2 - \gamma)^{-1}(b - a)/2$ is less than one, this shows that the equivalent map

$$w = (-\Delta - \gamma)^{-1})(I - P)((b - \gamma)(v + w)^+ - (a - \gamma)(v + w)^-)$$

is a contraction and this has a unique fixed point.

Finally, since either $v \leq 0$, or $v \geq 0$ in Ω, by explicit calculation we can check that $w \equiv 0$ is the unique solution. (This is because $v^+ = v$ and $(I - P)v = 0$ if $v > 0$.) This concludes the proof of the lemma.

Thus, we can see that if $a < \lambda_1 < b < \lambda_2$, the only solutions are the potential linear solutions. If $s < 0$, there is no solution, if $s = 0$, there is the (unique) zero solution, and if $s > 0$, there are exactly two solutions, one negative and one positive.

Almost twenty years after Dolph, it was shown that the piecewise linear model and the approximately linear solutions were a good model for the semilinear case. Part of what was proved was in [3] includes the following theorem.

THEOREM 2.4. *Let $f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2$ and $f''(s) > 0$ for all s. Then, if $h(x) = h_1 + s\phi_1$, for each $h_1 \perp \phi_1$, there exists a constant $C(h_1)$, such that if $s < C(h_1)$, (1) has no solution, if $s = C(h_1)$, (1) has one solution, and if $s > C(h_1)$, (1) has* exactly *two solutions.*

Actually, as we have stated it, the theorem is a combination of some of the ideas from [3], [8].

As we shall see later, the requirement that $f'(+\infty) < \lambda_2$ is necessary, if we wish to insist on *exactly* two solutions. However, at least in the piecewise linear case, existence of *at least* two solutions occurs as long as $a < \lambda_1 < b$.

Thus, one might conjecture that as long as $f'(-\infty) < \lambda_1 < f'(+\infty) < +\infty$, there would still exist at least two solutions for an unbounded interval of s. The first result in this direction was in [27], where it was shown that with these less restrictive hypotheses, there existed $C(h_1)$, such that if $s > C(h_1)$, there was *at least one* solution, and if $s < C(h_1)$, there was no solution.

This result clearly needed further clarification, which it received some years later in [4], [17], where it was shown that there were indeed at least two solutions if $s > C(h_1)$.

Somewhat later, the present authors observed [28] that, for large positive s, these solutions were approximately the two "linear-type" solutions, in the sense that they could be seen as perturbations of the piecewise linear solutions.

This summarizes the state of the "almost-linear" theory. The single-sign solutions of the piecewise linear theory are a good guide for finding lower bounds for the number of solutions of the semilinear equation: as long as convexity is maintained and one stays below λ_2, they are an exact guide.

It was natural to ask, then what additional phenomena could occur if, indeed, we do go beyond λ_2? This is the subject of the next section.

2.2. Asymmetric systems: Crossing the other eigenvalues.

This section, both from a historical and expository point of view, is divided into two parts. First, we consider the results where the nonlinearity f crosses the first several eigenvalues, and then the more complicated case where it crosses only higher ones.

2.2.1. Crossing the first several eigenvalues.

In [28], the authors first considered the equation

$$\Delta u + f(u) = h(x) + s\phi_1 \quad \text{in } \Omega, \quad u = 0 \text{ on } \delta\Omega \tag{6}$$

for large positive s, under the assumption that the nonlinearity f satisfied the condition

$$-\infty < f'(-\infty) < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3$$

or more generally,

$$-\infty < f'(-\infty) < \lambda_1 < \lambda_{2n} < f'(+\infty) < \lambda_{2n+1},$$

where, in addition, it was assumed that λ_2 was of odd multiplicity.

We showed that in this case, unlike the previous cases, (6) had *at least three*, and generically four, solutions.

We did this by showing that for large positive s, there existed approximately linear solutions, close to $s\phi_1/(f'(+\infty)-\lambda_1)$, and $s\phi_1/(f'(-\infty)-\lambda_1)$, respectively. Since these were almost solutions of linear equations, it was possible to calculate the Leray–Schauder degree of these solutions. The solution close to $s\phi_1/(f'(+\infty)-\lambda_1)$ had a topological degree of $(-1)^n$, where n was the number of eigenvalues, each counted as often as its multiplicity, in the interval $(f'(-\infty), f'(+\infty))$.

The solution close to $s\phi_1/(f'(-\infty)-\lambda_1)$, being below the first eigenvalue, had a topological degree of $+1$.

Then, it was shown that on a large ball, the topological degree $d_{LS}(0, I+\Delta^{-1}(f(u)-s\phi_1-h(x)), B_R)$ is zero. From this, by the usual excision properties of degree theory, if n is even, we concluded that the degree of the big ball, minus the two small balls centered around the two almost linear solutions was -2. This gave us the existence of at least one more solution and the generic existence, via Sard's theorem, of at least two solutions.

This showed, for example, in the piecewise linear case, that if $\lambda_2 < b < \lambda_3$ and λ_2 is simple, then there must exist additional solutions which *must change sign.* In the piecewise linear case, these were the first nonobvious solutions.

Naturally, this result raised more questions than it answered. One could ask if there were always four solutions, if $\lambda_2 < f'(+\infty) < \lambda_3$, or if this generic result was the best possible. Furthermore, this first result said nothing about the case $\lambda_{2n+1} < f'(+\infty) < \lambda_{2n+2}$, where degree theory (at least in its primitive form) tells us nothing.

The authors made a conjecture that if $-\infty < f'(-\infty) < \lambda_1 < \lambda_n < f'(+\infty) < \lambda_{n+1}$, there should exist at least $2n$ solutions. Although, ultimately, this conjecture proved false (at least for multiple eigenvalues) it inspired an sizable body of research, which we now briefly describe.

Before developing this theme, we should remark that any question that can be asked for elliptic partial differential equations can also be asked for the one dimensional case, namely, the ordinary differential equation. Naturally, we expect to prove more in this case, and we will return to this topic in the next subsection.

The outstanding question left by [28] was whether four solutions existed. This was soon answered in the affirmative in at least three independent ways.

The first method, and in the authors' opinion most impressive mathematically, was that of [24]. By a clever use of degree theory, Hofer built on the observations of [28], that there were two almost linear solutions. Both being almost solutions of the linear problem, their topological degrees could be calculated to be ± 1.

Hofer then showed that there must also exist a critical point which arises as a mountain pass, and that if it is isolated, then its topological degree must be -1. Now, excise a ball around each of the two almost linear solutions, and a ball around the mountain pass, from the big ball, and choose the radius of a big ball so large that the degree of the big ball is zero, and conclude that the remainder must have degree minus or plus one.

This powerful argument used only the fact that $-\infty < f'(-\infty) < \lambda_1$ and that $\lambda_n < f'(+\infty) < \lambda_{n+1}$ for some $n \geq 2$. One minor shortcoming of this result is that it was heavily dependent on the differential operator in the equation being the Laplacian.

A second avenue was used in [29]. Here, a purely operator-theoretic approach was used, relying only on the fact that there was an eigenvalue with positive eigenfunction, and one more simple eigenvalue that was crossed.

THEOREM 2.5. *Suppose $-\infty < f'(-\infty) < \lambda_1 < \lambda_2 < f'(+\infty) < \lambda_3$. Then, there exists $C^*(h)$ so that if $s \geq C^*$, the problem*

$$\Delta u + f(u) = h(x) + s\phi_1, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \delta\Omega \tag{7}$$

has at least four solutions. Moreover, if λ_3 is simple, there exists $\varepsilon > 0$ so that if s is sufficiently large, then (7) has at least five, and generically six, solutions.

Since it is possible to give an elegant operator theoretic proof of the main part of this theorem (the at least four solution part), we outline it here.

Proof (when $f(u) = bu^+ - au^-$). Let P be projection on the space spanned by ϕ_1, ϕ_2. Of course, $I - P$ is projection on the orthogonal complement. If $a < \lambda_1 < \lambda_2 < b < \lambda_3$, then as in Theorem 3, we can write $w = (I-P)u, v = Pu$, and $\Delta u + bu^+ - au^- = \phi_1$ is equivalent to

$$(I-P)\Delta w + (I-P)(b(v+w)^+ - a(v+w)^-) = 0, \tag{8}$$

$$(\Delta + \lambda_1)v + P((b-\lambda_1)(v+w)^+ - (a-\lambda_1)(v+w)^-) = s\phi_{-1}. \tag{9}$$

By the same arguments as before, we can observe that for each fixed v, there exists a unique solution $w(v)$ for (8). Moreover, by substitution into the equation, we can verify that if $v \geq 0$ or $v \leq 0$, then $w(v) \equiv 0$, and that $w(v)$ depends continuously on v.

Thus, we need only consider the two-dimensional map

$$\Phi(v) = \Delta v + P(b(v + w(v))^+ - a(v + w(v))^-), \tag{10}$$

and we ask if $\Phi(v) = \phi_1$.

A preliminary observation is that $\Phi(v)$ is never equal to $-s\phi_1$, since this would say that the equation

$$\Delta u + bu^+ - au^- = -s\phi_1$$

must have a solution, which, from our integration-by-parts trick, we know to be impossible.

By the same reasoning, $\Phi(v) = 0$ implies $v = 0$.

Now, choose R sufficiently large that $-R\phi_1 + \phi_2 \leq 0$ $R\phi_1 + \phi_2 \geq 0$, and consider $\{\Phi(v), v = t\phi_1 + \phi_2, -R \leq t \leq R\}$.

This is a curve in the two-dimensional PH space, which ends at the point $\Phi(R\phi_1 + \phi_2) = (b-\lambda_1)R\phi_1 + (b-\lambda_2)\phi_2$ (recall that $w(R\phi_1 + \phi_2 = 0)$ and starts at the point $\Phi(-R\phi_1 + \phi_2) = -(a-\lambda_1)R\phi_1 + (a-\lambda_2)\phi_2$.

Since the curve ends in the upper half plane, $c_1\phi_1 + c_2\phi_2, c_2 > 0$, and starts in the lower half plane, it follows that it must cross the ϕ_1-axis. That is, there exists a point on the axis such that $\Phi(t_0\phi_1 + \phi_2) = s^*\phi_1$ for some t_0, s^*.

By our earlier remarks, $s^* > 0$, and therefore, we can find a solution of

$$\Delta u + bu^+ - au^- = s^*\phi_1$$

with $s^* > 0$ and $(u, \phi_2) > 0$. Now, multiply across the equation by $1/s^*$. This solves our original problem. Similarly, we find a solution with $(u, \phi_2) < 0$. Therefore, counting the two linear solutions, there are at least four.

On seeing this result on more than four solutions, in [56] Solimini (who had already proved the existence of four solutions if the right-hand side was $(\phi_1 + \varepsilon\phi_2, \varepsilon \neq 0)$)

was able to show, under some additional technical hypotheses, that there were *exactly* four solutions for large s if $\lambda_2 < f'(+\infty) < \lambda_3$, and if $\lambda_3 < f'(+\infty) < \lambda_3 + \varepsilon$, then there were exactly six solutions.

This represents the state of progress of the problem of crossing the first n eigenvalues. It is known that if the second eigenvalue is a multiple eigenvalue, and is counted as often as its multiplicity, then it is possible to produce a region Ω, such that if $a < \lambda_1 < \lambda_2 = \lambda_3 < b$, then the equation

$$Lu + bu^+ - au^- = s\phi_1 + \varepsilon h(x)$$

has exactly four solutions, where in [34], L was an elliptic second-order operator and in [18] the Laplacian.

It may still be true, that in some generic sense, there are at least $2n$ solutions, but we appear to be far from a proof. On the other hand, what certainly is clear is that the principle enunciated in the introductory section is apparent in this context: *if we measure the asymmetry in the equation by the number of eigenvalues,* not counting the first, *of the Laplacian in the interval* $(f'(-\infty), f'(+\infty))$, *then the greater the asymmetry, the larger the number of oscillatory solutions.*

It is a curious fact that the first eigenvalue is distinguished by not creating any new oscillatory solutions, but only changing the distribution of the predictable ones.

2.2.2. Crossing several higher eigenvalues. Crossing the higher eigenvalues is what creates the oscillatory solutions. Therefore, it will not surprise the reader when we describe a similar family of results for crossing only some higher eigenvalues. One striking result for crossing higher eigenvalues appears in [32], where the following alternative theorem is proved.

THEOREM 2.6. *Suppose* $\lambda_n < f'(-\infty) = a < \lambda_{n+1}, \lambda_{n+k} < f'(+\infty) = b < \lambda_{n+k+1}, \lambda_n + \varepsilon < f' < \lambda_{n+k} - \varepsilon$. *Then one of the following two alternatives must hold:*

The single equation

$$\Delta u + bu^+ - au^- = 0$$

has an infinite number of solutions, or the two equations

$$\Delta u + f(u) = \pm\, s\phi_1 + h(x)$$

have a total of at least four solutions, for large positive s.

This second alternative may occur as (a) *three solutions for s large positive and one for s large negative,* (b) *two solutions for s large positive and two for s large negative, or* (c) *one solution for s large positive and three for s large negative. These numbers should be understood as lower bounds.*

The proof is variational in nature, and we will not give it here.

Earlier results, which did not use the forcing term ϕ_1, include [20], [46]. The new results are possible because of the combination of the positive forcing term and the crossing of the higher eigenvalues.

Again, we have the situation referred to in the opening section: if the two ingredients, a positive forcing term and asymmetry, are present, we expect multiple oscillatory solutions.

A result of [56] shows that if precisely one simple eigenvalue is crossed, then the estimates are precise. Of course, much more information is available in the case of the ordinary differential equation.

2.2.3. Open problems. There is a wealth of open problems remaining in this area. The most surprising simple question is about upper bounds. The next problem is based on the result of [3].

PROBLEM 1. *If $f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_3, f'' > 0$, then can there be* at most *four solutions to the equation for all s?*

It is known that if s is sufficiently large, then this is the case [56].

A second open question concerns the number of solutions if $f'(+\infty) = +\infty$, but f is allowed to grow slowly enough that compactness is maintained. We know that for any f satisfying $f'(+\infty) < +\infty$, there are at least four.

PROBLEM 2. *If $f'(-\infty) < \lambda_1 < f'(+\infty) = +\infty$, with appropriate growth restrictions, are there at least four solutions?... (much harder) Does the number of solutions become unbounded as s goes to $+\infty$?*

We can ask similar questions about crossing the higher eigenvalues. Again, apart from [56], almost nothing is known.

PROBLEM 3. *If $\lambda_n < f'(-\infty) < \lambda_{n+1}, \lambda_{n+1} < f'(+\infty) < \lambda_{n+2}, f'' > 0$, can we say that for all s, the equation has at most three solutions?*

2.2.4. A quick look at "elliptic-like" problems. For some time, it has been part of the folklore in semilinear elliptic problems that if you replaced the Laplacian with a wave operator or with a parabolic operator, then many of the results would continue to hold [11]. For example, if we study

$$\begin{aligned} &u_{tt} - u_{xx} + f(u) = h(x,t), \\ &u(0,t) = u(\pi,t) = 0, \\ &u(x,t+2\pi) = u(x,t), \end{aligned} \tag{11}$$

then with an additional assumption of monotonicity of f in the variable u, many of the standard results do go through [11]. It is natural, therefore, to wonder which of the above results can be extended to this setting.

One of the problems is that most of the results rely heavily on elliptic properties such as the maximum principle and eigenvalue comparison theorems. These simply will not work in this new setting. Some results, however, have been obtained.

The wave operator has spectrum $\{(n^2 - m^2, n \geq 1, m \geq 0\}$. The eigenvalue 1 has multiplicity one with the positive eigenfunction, $\sin(x)$.

In [40], it was shown that if $h(x,t) = s\sin(x) + h_1(x,t)$, then (11) has at least two solutions if $0 < f'(-\infty) < 1$, and $1 < f'(+\infty) < 3$, *and f is monotone increasing.*[3]

One interesting point is that under these conditions on f, the equation

$$-u_{xx} + f(u) = s\sin(x), \qquad u(0) = u(\pi) = 0 \tag{12}$$

has a unique solution, by the usual Hammerstein method. Thus, there is a natural time-independent unique steady-state solution to (11). Furthermore, it is easy to check that it is like $s\sin(x)/(f'(+\infty) - 1)$.

Therefore, the additional solution created must by its very nature be oscillatory. The asymmetric restoring term has created oscillatory phenomena.

Much of this area remains unexplored. For example, to the authors' knowledge, there are no results on crossing higher eigenvalues using critical point theory.

[3] There are some minor symmetry restrictions on $h_1(x,t)$.

Similarly, we can look for multiplicity results for the semilinear parabolic problem,

$$\begin{aligned} &u_t - u_{xx} + f(u) = s\sin(x) + h(x,t) \\ &u(0,t) = u(\pi,t) = 0 \\ &u(x,t+T) = u(x,t). \end{aligned} \tag{13}$$

Some results are contained in [29] and [40].

Perhaps similar results can be obtained for the telegraph equation, where results similar to those of Dolph have been discovered [38].

2.3. The one-dimensional boundary value problem. Of course, all the results that we have described for elliptic partial differential equations are true for the one-dimensional Dirichlet problem. It is natural to ask what additional information can be obtained, using the powerful additional tools available from ordinary differential equations.

Motivated by the short-lived conjecture that crossing the first n eigenvalues would create $2n$ solutions, the authors studied in [30] the equation,

$$u'' + f(u) = s\sin(x) + h_1(x), \qquad u(0) = u(\pi) = 0. \tag{14}$$

The following theorem was proved.

THEOREM 2.7. *Let f satisfy $f'(-\infty) < 1 < n^2 < f'(+\infty) < (n+1)^2$. Then for $2s \gg 1$, (14) has at least $2n$ solutions.*

Proof (sketch). For large s, we expect the almost linear solutions $v^* = s\sin(x)/(f'(+\infty) - 1)$ and $v_* = s\sin(x)/(f'(-\infty) - 1)$. Now, centering around the large positive solution v^*, we let $u = v^* + w$.

Then w satisfies

$$w'' + f(v^* + w) - f(v^*) = 0, \qquad w(0) = w(\pi) = 0. \tag{15}$$

We expect that by solving the initial value problem, $w_\varepsilon(0) = 0, w'_\varepsilon(0) = \varepsilon$, we will find that $w_\varepsilon$will behave much like the solution to the linear problem $\hat{w}'' + f'(+\infty)\hat{w} = 0$.

Now, let $R = v'_*(0) - v^*(0)$. Then we know that the initial value problem $w''_R + f(v^* + w_R) - f(v^*) = 0, w_R(0) = 0, w'_R(0) = R$, will look like $v_* - v^*$, which has no zero in the interval $(0,\pi)$. Putting these two facts together, we can use the well-known shooting technique to prove the existence of the $2n$ solutions by keeping careful track of the number of zeros of the initial value problem $w''_r + f(v^* + w_r) - f(v^*) = 0, w_r(0) = 0, w'_r(0) = r$, as r varies.

PROBLEM 4. *Assuming that f is convex and that $f'(-\infty) < 1 < n^2 < f'(+\infty) < (n+1)^2$, can one give* upper bounds *on the number of solutions of* (14)*?*

We expect that *for all s*, (14) has at most $2n$ solutions. Amazingly, this is not even known in the case $n = 2$, unless $s \gg 1$.

We can also ask the question of Problem 2 in this context, when $f'(+\infty) = +\infty$. As we increase $f'(+\infty)$ and it remains finite, we get more and more solutions. It is natural to conjecture that the following statement is true: Suppose that $f'(-\infty) < 1 < f'(+\infty) = \infty$. Then, for all N, there exists S_N such that if $s > S_N$, (14) has at least $2N$ solutions.

There has been some limited progress on this conjecture. Usually, the results seem to rely a very specific nonlinearity $f(u)$, say either u^2 or $(u^+)^2$, and $h(x) = 0$ (see [15], [49]). The difficulty seems to be that there is no equivalent to the large almost linear positive solution v^* around which to center for a successful shooting.

Again the question arises as to what happens when only the higher eigenvalues are crossed. In [23], the following was proved.

THEOREM 2.8. *If* $n^2 < f'(-\infty) < (n+1)^2$ *and* $(n+k)^2 < f'(+\infty) < (n+k+1)^2$, *then the total number of solutions of* (14), *for* $s \gg 1$ *and* $s \ll -1$ *is at least* $2k+2$, *assuming the homogeneous problem*

$$u'' + f'(+\infty)u^+ - f'(-\infty)u^- = 0, \qquad u(0) = u(\pi) = 0,$$

has no nontrivial solution.

Actually, the theorem said a good deal more, giving additional information on precisely how many solutions occurred for large positive s and how many occurred for large negative s.

Again, the paper [23] said nothing about the case where $n^2 < f'(-\infty) < (n+1)^2$ and $f'(+\infty) = +\infty$. We are tempted to conjecture, based on phase plane analysis for the corresponding Neumann problem, that in this case there should be at least n solutions if s is large negative, and an increasing unbounded number as s becomes large positive.

Some progress has been made on the first of these two cases (again using special nonlinearities and $h(x) = 0$), none on the second half. Again the problem seems to be that there is no natural positive solution about which to center and shoot.

Again, we have no information here about upper bound, except in the very limited circumstances of [56].

PROBLEM 5. *If* $n^2 < f'(-\infty) < (n+1)^2$, $(n+1)^2 < f'(+\infty) < (n+2)^2$, *and* $f''(s) > 0$, *are there always at most three solutions to* (14)*? ...(much harder) If* $n^2 < f'(-\infty) < (n+1)^2$, $(n+k)^2 < f'(+\infty) < (n+k+1)^2$, *and* $f''(s) > 0$, *are there at most* $2k+1$*?*

2.4. A short summary on the Dirichlet problem. All of these results have a common theme. First, a large positive right-hand side gives rise to an obvious almost linear solution. If there is not much asymmetry in the equation, in the sense that no eigenvalues are crossed, then this solution is unique. On the other hand, if eigenvalues are crossed, large numbers of oscillatory solutions result.

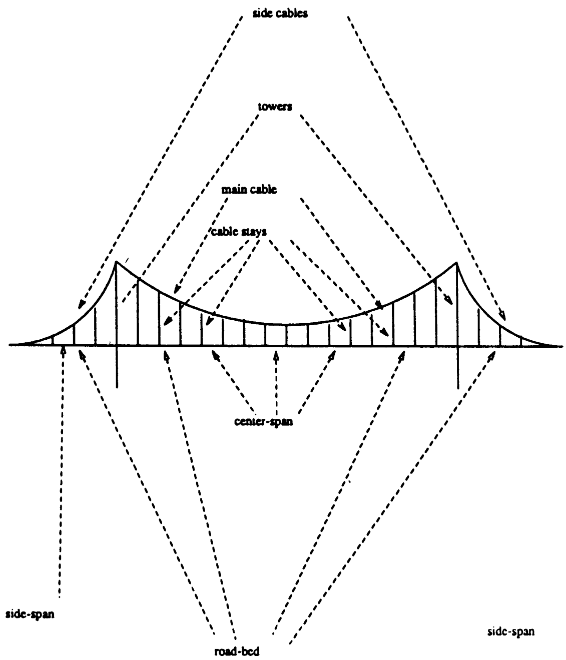
For the one-dimensional boundary value problem, good lower bounds for the number of solutions have been established. In the case of the nonlinearity crossing the bottom eigenvalues, solutions only exist for positive multiples of the first eigenfunction. In the case of crossing higher eigenvalues, solutions may exist in the positive and negative directions. Even in this situation, information on the *exact* number of solutions is hard to come by.

In the case where the nonlinearity grows more rapidly than linearly, there is little information, except for special cases.

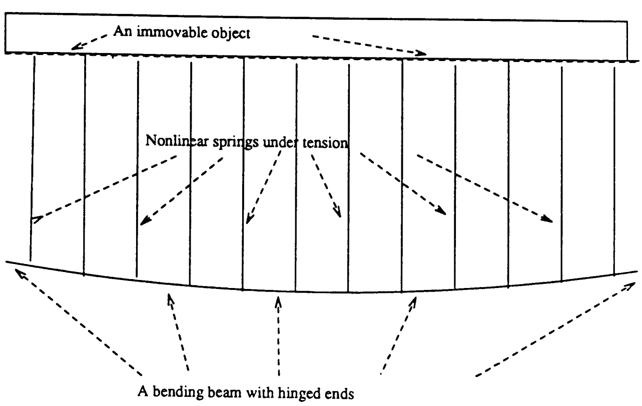
For the case of more than one space dimension, the results are far less complete for the elliptic problem. Lower bounds on the number of solutions indicate that situation is similar to that of the one-dimensional problem, but the possibility of multiple eigenvalues makes the results more difficult.

3. Back to the suspension bridge. We now return to the problem of nonlinear oscillation in a suspension bridge. We have seen that a linear model is insufficient to explain the large oscillatory behavior that has been observed. In addition, suspension bridges are known to have other nonlinear behaviors such as travelling waves [5], [41].

In the first section, we will write down the simplest nonlinear partial differential equation that we can, which takes account of the fact that the stays connecting the



(a)



(b)

FIG. 2. (a) *The main ingredients in a one-dimensional suspension bridge.* (b) *The first idealization of the suspension bridge: the beam bending under its own weight is supported by the nonlinear cables. Motion of the cables will be treated as an external forcing term on the beam.*

cable to the deck of the bridge are fundamentally nonlinear, in that if you pull on a rope, it resists, whereas if you push, it does not. We shall treat the stays as one-sided springs, obeying Hooke's law, with a restoring force proportional to the displacement from the unstretched state if stretched, and with no restoring force if compressed (see Fig. 2).

The roadbed will be treated as a one-dimensional vibrating beam [55]. The motion of the cable will be ignored except insofar as the stays transmit a forcing term to the roadbed. This gives rise to the following equation:

$$(16) \qquad \begin{aligned} &u_{tt} + EIu_{xxxx} + \delta u_t = -ku^+ + W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \end{aligned}$$

Thus the suspension bridge is seen as a beam of length L, with hinged ends, whose downward deflection is measured by $u(x,t)$, with a small viscous damping term, subject to three separate forces; the stays, holding it up as nonlinear springs with spring constant k, the weight per unit length of the bridge $W(x)$ pushing it down, and the external forcing term $\varepsilon f(x,t)$, about whose origin we will not comment until later, but which we will assume to be periodic. The loading $W(x)$ would usually be constant.

We emphasize at this point that we do not believe this completely models the complex behavior of the bridge. However, if this simple model exhibits unexpected complex oscillatory behavior, then a more accurate model can reasonably be expected to do so.

Normally, the suspension bridge will be close to the equilibrium position, given as the solution of the steady state equation

$$(17) \qquad \begin{aligned} &EIy^{(4)} + ky^+ = W(x), \\ &y(0) = y(L) = y''(0) = y''(L) = 0. \end{aligned}$$

If we have a small periodic forcing term $\varepsilon f(x,t)$, we expect to find a periodic solution of (16), which is close to equilibrium. This will be a solution of the *linear* equation

$$(18) \qquad \begin{aligned} &u_{tt} + EIu_{xxxx} + \delta u_t + ku = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0. \end{aligned}$$

If we are studying small-amplitude solutions, we expect the linear model to give good agreement with the experimental data. The question, of course, is whether (16) has any other solutions.

3.1. Periodic solutions of a bridge-like ordinary differential equation. We start the discussion of (16) with an oversimplification. Instead of taking the weight of the bridge to be constant, we replace it by the first term in the eigenfunction expansion of the constant function; that is, we replace W by the term $W(x) = W_0 \sin(\pi x/L)$. This introduces an error of magnitude 10% in the loading and somewhat less in the steady-state deflection.

Second, we assume the forcing term is given by $f(x,t) = f(t)\sin(\pi x/L)$. This is a peculiar term, but there is no reason why the bridge cannot have this type of forcing term.

Finally, instead of looking for general solutions of (16), we look for no-node solutions of the form $u(x,t) = y(t)\sin(\pi x/L)$. (These no-nodal solutions were the most commonly observed type for low velocities on the Tacoma Narrows Bridge [5].)

When this $u(x,t)$ is substituted into (16), we can take the term $\sin(\pi x/L)$ out of the nonlinearity, and divide across by it. When we do this, we obtain

$$(19) \qquad y'' + \delta y' + EI(\pi/L)^4 y + ky^+ = W_0 + \varepsilon f(t).$$

If we let $b = EI(\pi/L)^4 + k$ and $a = EI(\pi/L)^4$, we obtain (with the exception of the small damping term) the type of equation studied as a boundary value problem in §2.

Thus, we are led to consider the periodic solutions of the problem

$$
\begin{aligned}
&y'' + f(y) = c + g(t),\\
&y(0) = y(2\pi),\\
&y'(0) = y'(2\pi),
\end{aligned}
\tag{20}
$$

with $f'(+\infty) = b$ and $f'(-\infty) = a$. The constant c is a multiple of the first eigenfunction, and g is periodic. We ask if there are multiple solutions when there is a large gap between a and b. From the earlier results, we expect a large number of solutions as the difference between a and b increase, and indeed this proves to be the case.

The first of these theorems was proved in [33].

THEOREM 3.1. *Let N_+ be the number of integers j, such that $1/\sqrt{a} + 1/\sqrt{b} > 2/j > 2/\sqrt{b}$. Then for large positive c,* (20) *has at least $2N_+ + 1$ solutions. Let N_- be the number of integers j, such that $1/\sqrt{a} + 1/\sqrt{b} < 2/j < 2/\sqrt{a}$. Then for large negative c, the* (20) *has at least $2N_- + 1$ solutions.*

Assume that $y'' + by^+ - ay^- = 0$ has no nontrivial 2π-periodic solutions. Note that $N_+ + N_- = 2k + 2$, where k is the number of eigenvalues crossed. Let us look at this result more closely for the piecewise linear bridge-like equation.

$$y'' + by^+ - ay_- = 1 + \varepsilon g(t). \tag{21}$$

If $b \neq n^2$, we can explicitly write down a 2π-periodic solution of this equation, namely, $1/b + \varepsilon y_1(t)$, where y_1 is the 2π-periodic solution of $y'' + by = g(t)$. This is the *physically obvious solution.* A large push $c = 1$ in (21), plus a small vibrating term $\varepsilon g(t)$ produces a large displacement $1/b$ plus a small oscillation about the new equilibrium of order of magnitude ε. Moreover, by the usual Dolph-type argument, if $n^2 < a,\ b < (n+1)^2$, then this is a unique solution of period 2π.

Theorem 3.1 is less intuitively obvious. It says that if the difference between a and b is large, then additional numbers of oscillatory solutions exist, *and their order of magnitude is that of c.*

This is the beginning of the theory we felt was required to explain the large oscillations of the Tacoma Narrows Bridge. The subject arises naturally from the suspension bridge, and only requires that the relevant parameters fall in intervals. The intervals get larger as the bridge relies more on the spring constant k and less on the rigidity of the deck.

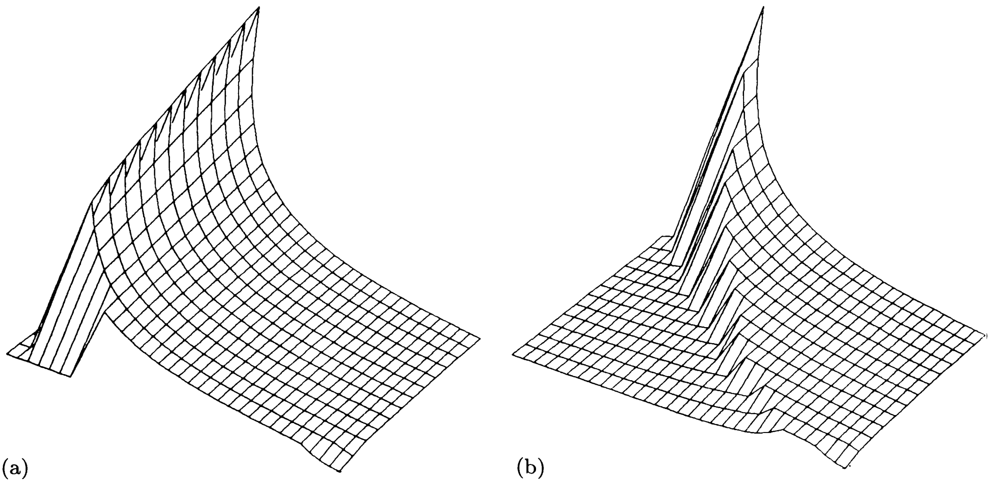
This result raises more questions than it answers. For example, it says nothing about what happens if there is small damping. Indeed, if we take $\varepsilon = 0$, then the introduction of small damping destroys the large amplitude solutions.

If suitable conditions on $g(t)$ are imposed to guarantee that the large amplitude solutions persist in the presence of damping, we can then ask whether they are stable. In addition, we can ask whether, with large amplitude initial data, the solution of the initial value problem converges to the large amplitude periodic solution or the solution close to steady-state.

These questions were partially answered in [21]. If we look for T-periodic solutions of

$$u'' + bu^+ - au^- = 1 \tag{22}$$

with $2\pi/\sqrt{b} < T < \pi/\sqrt{b} + \pi/\sqrt{a}$, then we can find, by simple phase-plane analysis, a large amplitude solution of order of magnitude one, and of least period T which we



(a) (b)

FIG. 3. (a) *The large amplitude solution reached by suitable choice of large initial conditions.* (b) *The results of starting near the "approximately linear" solution.*

denote by u_0. It was shown in [21], that with a mild nondegeneracy condition on $g(t)$, with ε and c small, then there exist large amplitude solutions of

$$y'' + \varepsilon c y' + b y^+ - a y^- = 1 + \varepsilon g(t), \tag{23}$$

which are asymptotically stable and close to a translate of u_0.

This, in turn, leads to the question of whether these solutions arise naturally in numerical solutions of the initial value problem. Extensive computation has shown, as detailed in [21], that when viewed as a two parameter family, the solutions of

$$y'' + 0.01 y' + 17 y^+ - 13 y^- = 10 + \lambda \sin(\mu t) \tag{24}$$

behaved like a cusp, with a lower surface (the approximately linear solutions) and an asymptotically stable upper surface. The focal point of the cusp appears at the point of linear resonance, $\mu = \sqrt{17}$.

If we start with very small initial data, we stay on the lower surface until some critical value of amplitude $\lambda_0(\mu)$, and then jump to the upper surface. Computation then showed that that as λ becomes large the solution again becomes unique. In [35], this was proved analytically.

The two surfaces are visible in Fig. 3, where the amplitude of the periodic solution converged to in large time, as computed numerically, is shown with either large or small initial data. The surface then becomes infinite, apparently, when $2/\mu = 1/\sqrt{17} + 1/\sqrt{13}$. If the constant 13 is replaced by the constant 1 (making the bridge more flexible and relying more on the cable) we find that the interval of multiple period solutions increases.

This summarizes the state of knowledge of the piecewise linear case. Perhaps the most interesting open problem is to describe in analytic terms the properties of the cusp-like surface that is revealed in computation. Although there have been some results of this type for the boundary value problem [57], [6], [49], we know of no results in the periodic setting.

3.2. Results for the partial differential equation. We now study the partial differential equation

$$(25) \qquad \begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t = -ku^+ + W_0 + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \end{aligned}$$

which we approximated in the earlier subsection. As we saw, this represents the bridge as a vibrating beam of length L, with hinged ends, supported on one side by nonlinear springs with spring constant k. The springs are in a state of tension, due to the weight of the roadbed, which is now taken to be W_0 per unit length. This is represented in Fig. 2. Needless to say, the results are not as plentiful as for the ordinary differential equation case.

In [39], the following result was proved.

THEOREM 3.2. *Let $\delta = 0, L = \pi$, and $T = \pi$. In addition, let $f(x,t)$ be even in t, T-periodic in t, and even in x about $\pi/2$. Then, if $0 < k < 3$, (25) has a unique periodic solution of periodic π (the physically obvious one of small oscillation about the equilibrium).* However, if $3 < k < 15$, the equation had, in addition, a large amplitude periodic solution.

This can be summarized as, "*Strengthening a bridge can lead to its destruction*," in the sense that strengthening the stays can increase k. It is a curious fact that this was one of the first alterations proposed and put into effect for the Tacoma Narrows Bridge in the futile efforts to modify its dynamic behavior before its eventual collapse. Needless to say, it did not have the desired effect.

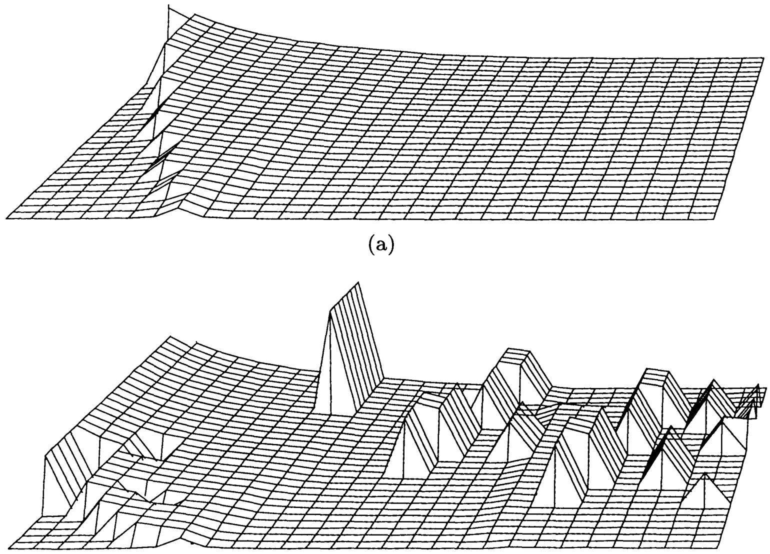
PROBLEM 6. *Is there a nondegeneracy condition on $f(x,t)$, which will ensure that solutions of* (25) *persist if damping is present?*

PROBLEM 7. *Can the restriction that $k < 15$ be removed? When comparing this theorem with the corresponding* ODE *result, it seems clear that increasing k still further ought to* increase *rather than decrease the number of solutions. (This restriction is a limitation of the method of proof.)*

In search of further information about solutions of (25), we solved the equation numerically. We used a finite difference method, implicit in the linear part, and explicit in the nonlinear part. The method proved stable under the usual precautions, such as halving stepsizes, and comparing results. Since we were using a large amount of CPU-time, most of the results described here were done with a stepsize in x of 0.1, and a stepsize in t of 0.05.

Equation (25) was solved for various lengths of the bridge with $\delta = 0.01, EI = 1, k = 18, W = 10$, with various forcing terms of the general form $\lambda \sin(\mu t)\sin(n\pi x/L)$ and various initial conditions of either large or small amplitude were used. In this way, we hoped that if there were large amplitude periodic solutions around, we would converge to them in large time. This plan worked, and showed that there were indeed multiple solutions.

For a short bridge, $L = 3$, the ordinary differential approximation was an excellent model. If we took $n = 1$, the no-node forcing term, we could indeed expect to converge in large time, to different no-node periodic solutions over a wide range of λ and μ. Figure 4 shows the amplitude of the eventual solution as a function of λ and μ, with either large or small initial conditions. The forcing term is $\lambda \sin \mu t \sin \pi x/L$ with $L = 3.0$. The frequency μ varies from left to right, from $\mu = 3.0$ to $\mu = 8.4$, and λ varies from 0.0 to 6.0. Resonance is clearly visible for small values of λ, near $\mu = 4.4$. Below resonance, it is clear that there are multiple solutions. The spikes on the



(a)

(b)

FIG. 4. *The magnitude of the finite difference solution of the partial differential equation after large time with either* (a) *large initial values, or* (b) *small initial values.*

right of resonance indicate that the high frequency perturbation has given rise to a half-frequency large-amplitude solution.

For a longer bridge, $L = 6$, and all other constants the same, the same phenomena were observed. However, even with a symmetric forcing term, sometimes the large amplitude symmetric solution appeared to become unstable, and some form of symmetry-breaking occurred, resulting in convergence to solutions which oscillated at one end; see Fig. 5. Some slight asymmetry is introduced into the system by the roundoff errors in π, which was deliberately left as 3.14159.

Another intriguing numerical result was convergence to a solution that appeared to be a wave, travelling up and down the bridge, and being reflected at the end-points.

Possibly, in a two-space dimensional model, the large amplitude solution would become unstable in a two-dimensional way, perhaps giving rise to the torsional motion that arose in the Tacoma Narrows Bridge, when the amplitude of the one-dimensional motions became large in the extremely violent storm that destroyed it.

PROBLEM 8. *Can any stability results be established for the large-amplitude solutions?*

PROBLEM 9. *Can we give a rigorous proof of the existence of unsymmetric solutions of* (25) *in the case where* $\delta = 0$ *and* $\varepsilon = 0$*?*

3.3. Travelling waves at the Golden Gate Bridge. One of the most interesting nonlinear phenomena mentioned in [5] was not the oscillation of the Tacoma Narrows Bridge, but the appearance of travelling waves on the Golden Gate Bridge. During an unusually violent storm on the night of Feb. 9, 1938, Mr. R. G. Cone, the chief engineer of the bridge, reported:

> The force of the wind was so strong that it was impossible to stand erect on the sidewalk, or the roadway of the bridge.... I observed that the suspended structure of the bridge was undulating vertically in a wavelike motion of considerable amplitude ...the wave motion

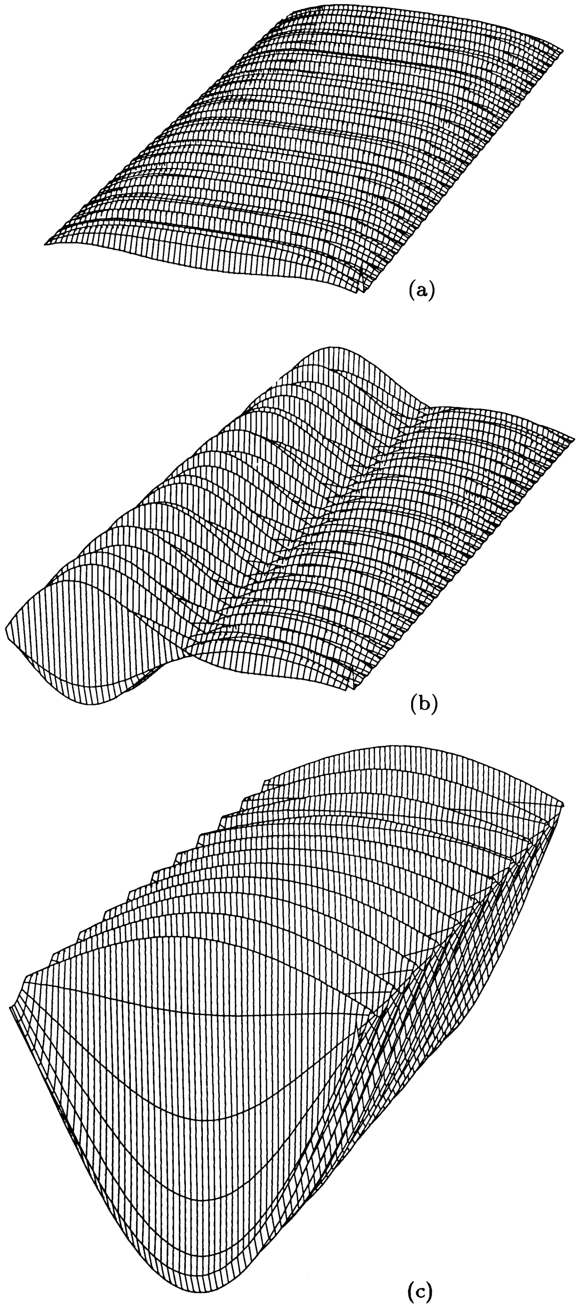


FIG. 5. *Multiple solutions of* (25) *with constant values and forcing term as described in* §3.2, *with* $n = 1$ *and* μ *and* λ *given.*

> appeared to be similar to that made by cracking a whip. The truss would be quiet for a second, and then in the distance, one could see a running wave of several nodes approaching.... the oscillations and deflections of the bridge were so pronounced that they would seem unbelievable. (pp. ix-1)

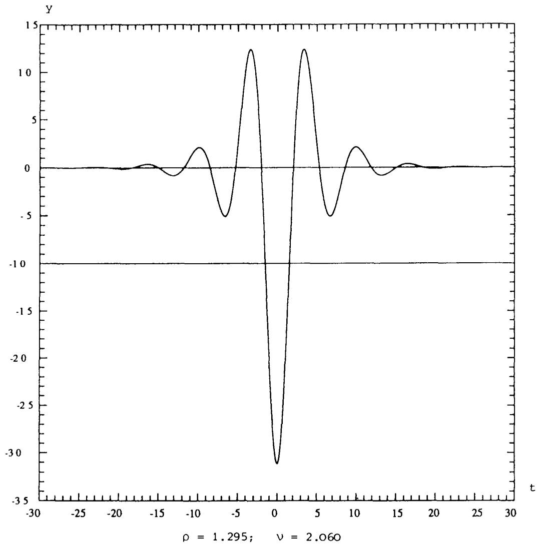


FIG. 6. *A likely candidate for the travelling wave as described by R. G. Cone on the Golden Gate Bridge in* 1938. *"In the distance, one could see a running wave of several nodes approaching...the oscillations and deflections of the bridge were so pronounced that they would seem unbelievable."*

Although the oscillation continued for some time, when Mr. Cone eventually returned with a camera, it had died down. A good test of any nonlinear model is whether it can show this type of behavior. In [41], (25) was assumed to hold on an infinite beam, and it was shown that there existed large families of travelling waves. Figure 6 shows a likely candidate for the waves described above.

The solution shown in Fig. 6 was obtained by finding solutions of the

$$u_{tt} + u_{xxxx} = -ku^{+} + W_0$$

on the real line, requiring the function $y(x)$ to tend to zero exponentially fast as x tends to infinity. Since finding solutions of this form requires solving an ordinary differential equation, solutions are found explicitly. By suitable normalization, we can take $k = 1$ and $W = 1$. Then $W \equiv 1$ is an equilibrium.

We look for solutions of the form $u(x,t) = 1 + y(x - ct)$ so the function y satisfies

$$y'''' + c^2 y'' + (y+1)^{+} - 1 = 0$$

on the real line, decaying exponentially at infinity. As shown in [41], this involves solving the two linear second order equations $y'''' + c^2 y'' = 1$ for $y < -1$, and $y'''' + c^2 y + y = 0$ for $y \geq -1$ and matching them at $y = 1$. It is, in essence, an extremely difficult second-year undergraduate calculus problem.

PROBLEM 10. *Show that the travelling waves whose existence was demonstrated in* [41] *are stable. In addition, show that if the nonlinearity* $-ku^{+}$ *is replaced by a somewhat more general function, the solutions persist. More generally, find properties of these families of travelling waves, including their interactions.*

PROBLEM 11. *Give a variational or other functional-analytic proof the solutions* y *exist. Try to generalize to more than one space dimension for the partial differential equation*

$$\Delta^2 u + c^2 \Delta u + (y+1)^{+} - 1 = 0 \quad \text{on } R^n$$

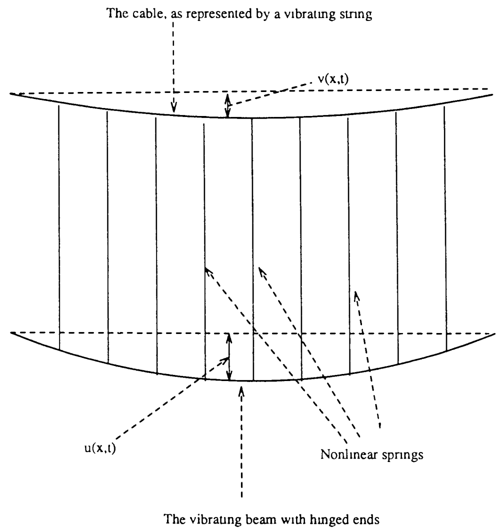


FIG. 7. *A more complicated model of the one-dimensional bridge represents the coupling of the cable (a vibrating spring) and the roadbed (a vibrating beam) by the stays, treated as nonlinear springs (see* (26)).

or (better), do the same for a more general f which is "like" $(y+1)^+ - 1$.

3.4. Coupling the motions of the roadbed and the cable. It is unnatural to ignore the motion of the cable in this study. In this section, we will make a very brief beginning into an area that promises to bear much fruit in the future.

We treat the cable as a vibrating string, coupled with the vibrating beam of the roadbed by nonlinear springs that have a spring constant k, if expanded, but no restoring force if compressed. The beam is subject to its weight, and the cable is subject to some oscillatory forcing term which might be due to the wind or to motions in the towers or side-spans. This idealization is shown in Fig. 7.

Then we study

$$\begin{aligned} &m_1 v_{tt} - T v_{xx} + \delta_1 v_t - k(u-v)^+ = \varepsilon f_1(x,t), \\ &m_2 u_{tt} + EI u_{xxxx} + \delta_2 u_t + k(u-v)^+ = W_1, \\ &u(0) = u(L,t) = u_{xx}(0) = u_{xx}(L,t) = v(0) = v(L,t) = 0. \end{aligned}$$

The primary difference between this system and the model used in the classical engineering literature [7] is that in the engineering literature, the stays connecting the roadbed to the supporting cable are treated as inextensible rods, incapable of either extension or compression. This allows the motion of the cable and the roadbed to be treated as a single equation, but is clearly inappropriate when considering the large scale oscillations in which the stays are known to alternately loosen and tighten [5].

The mass per unit length of the cable, m_1, is much less than the mass per unit length of the roadbed. If we divide across by the respective masses, we get the system of equations

$$\begin{aligned} &v_{tt} - c_1 v_{xx} + \delta_1 v_t - k_1(u-v)^+ = \varepsilon f(x,t), \\ &u_{tt} + c_2 u_{xxxx} + \delta_2 u_t + k_2(u-v)^+ = W_0, \\ &u(0) = u(L,t) = u_{xx}(0) = u_{xx}(L,t) = v(0) = v(L,t) = 0. \end{aligned} \tag{26}$$

Here, the variable v measures the displacement from equilibrium of the cable and the variable u measures the displacement of the beam. Both are measured in the downward direction. The stays connecting the beam and the string act so as to pull the cable down, hence the minus sign in the first equation, and to hold the roadbed up, therefore causing a plus sign in the second.

Since, in both equations, we have divided across by the mass per unit length of the bridge, we expect that k_2 will be an order of magnitude smaller than k_1. The constants c_1 and c_2 represent the relative strengths of the cable and the roadbed.

Little work has been done on (26). We expect that it will prove just as rich in interesting phenomena as the earlier single-equation model.

However, in the spirit of §3.1, we could use the earlier approximation and replace the constant term in the second equation by the first term in its eigenfunction expansion, thus giving us a right-hand side $W_0(x) = W_0 \sin(\pi x/L)$ in the second equation of (26).

This was a good approximation for the simpler phenomena of the bridge of length three, so we expect it to be a similarly good guide in this setting.

Again, we look for no-node solutions of the form $u(x,t) = y(t)\sin(\pi x/L)$ and $v(x,t) = \sin(\pi x/L)$, and impose a forcing term of the form $f(x,t) = g(t)\sin(\pi x/L)$. After the same manipulations as before, this leads to an equation of the form

$$\begin{aligned} z'' + \delta_1 z' + a_1 z - k_1(y-z)^+ &= \varepsilon g(t), \\ y'' + \delta_2 y' + a_2 y + k_2(y-z)^+ &= W_0. \end{aligned} \tag{27}$$

This simple and innocent-looking equation has some very interesting properties that we are just beginning to explore. There are some preliminary results, theoretical and numerical, which we will briefly describe.

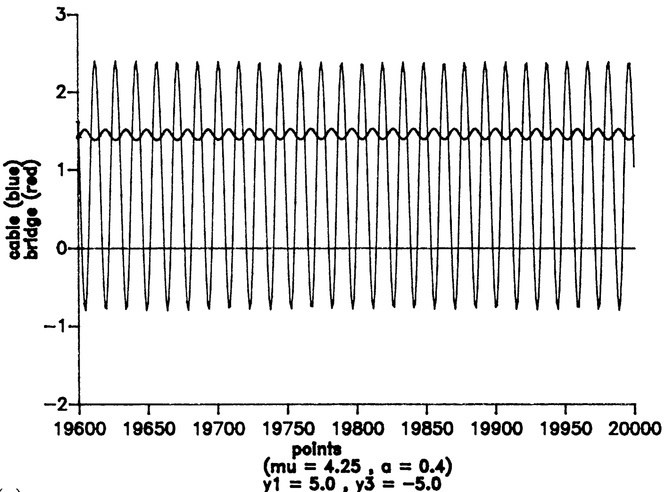
First, the theoretical results. We have studied the undamped case $\delta = 0$, under the additional assumption that k_2 is small. In this case, we were able to show that (27) has, for sufficiently small ε and k_2, large and small amplitude periodic solutions over a wide range of amplitude and frequency. The large amplitude solutions had y, the motion of the roadbed, close to equilibrium, and z, the motion of the cable, large.

Thus, we are led to predict that for this type of bridge, there exists the phenomenon of *galloping cables.* What else can happen is something that has yet to be explored, although we do have some numerical evidence.

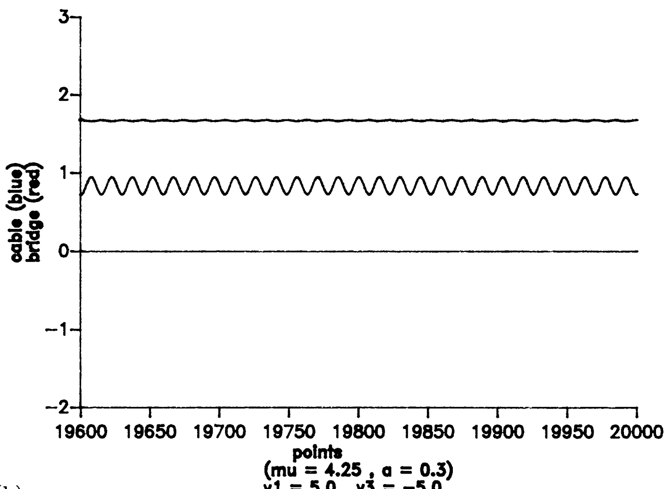
Second, the numerical results. We solved system (27), using a standard IMSL subroutine on a mainframe using high precision, with the constants taken as $a_1 = 10$, $a_2 = 0.1$, $\delta_1 = \delta_2 = 0.01$, $k_1 = 10.0$, and $k_2 = 1.0$. The weight W_0 was one and the forcing term $g((t) = \lambda \sin(\mu t)$ and we looked for periodic solutions for various values of λ and μ. These values were supposed to represent a highly flexible bridge with a large (relatively speaking) weight per unit length, and a strong cable.

Figure 8 shows two of the interesting phenomena that we found. As predicted by the theoretical result mentioned above, there was indeed the phenomenon of galloping. With $\mu = 4.25$, as λ varies from 0.3 to 0.4, we find two different periodic solutions, one of large amplitude, and one of small amplitude. In both, the bridge is barely moving.

What happens when λ increase further, to the point where the bridge is moving more violently? Now, we take μ equal to 4.5. Figure 9 shows how, when λ increases from 2.3 to 2.4, we get a different large-scale motion, in which the bridge and the cable are coupled. The cable appears to be in a beat-like oscillation and the bridge



(a)



(b)

FIG. 8. *Nonlinear behavior in the numerical solutions of the simplified cable-bridge equation* (27).

is moving in a large period motion. As we increase λ even further to 3.0, a different periodic motion appears, in which the cable appears to be driven by the bridge, but is also oscillating at the frequency of the forcing term.

This suggests an entirely new mechanism to explain why a suspension bridge would oscillate in a violent storm. First, the gusts of wind would act as a random large buffeting force on the cable superstructure, causing the towers and cable to go into a high frequency periodic motion (much as what happens when a guitar string is struck randomly). Then, as suggested in Fig. 9, nonlinear coupling would take place, and the bridge would go into a low frequency motion.

This does not address the torsional oscillation which was eventually responsible for the destruction of the Tacoma Narrows Bridge, but provides a reasonable explanation of the vertical modes of oscillation which were most commonly observed on this bridge. In the next section, we shall make some preliminary comments on the torsional oscillations.

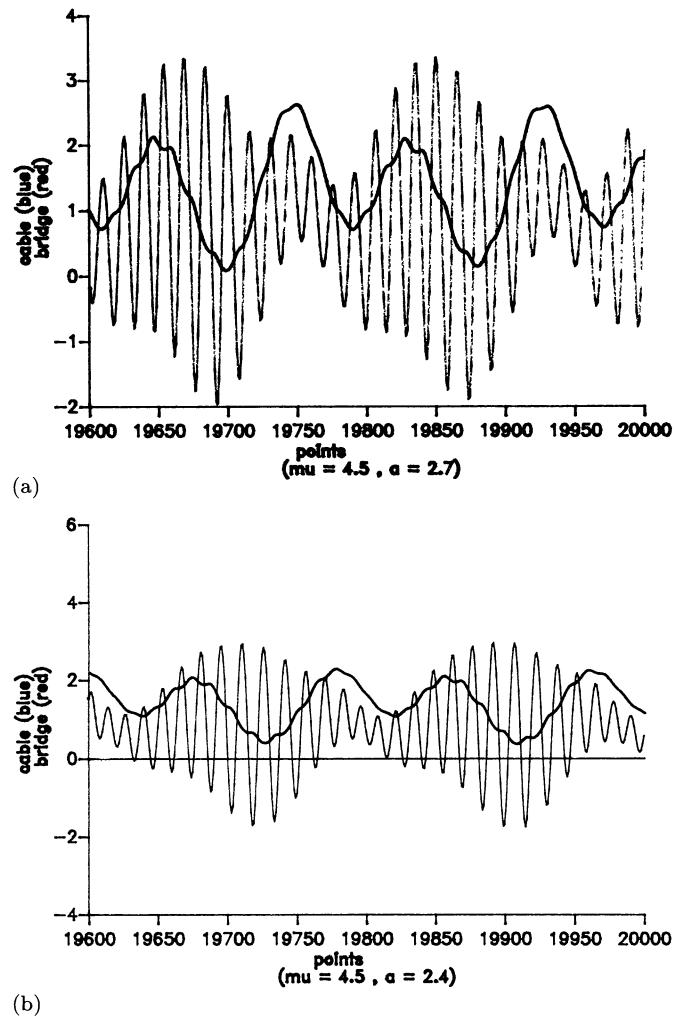
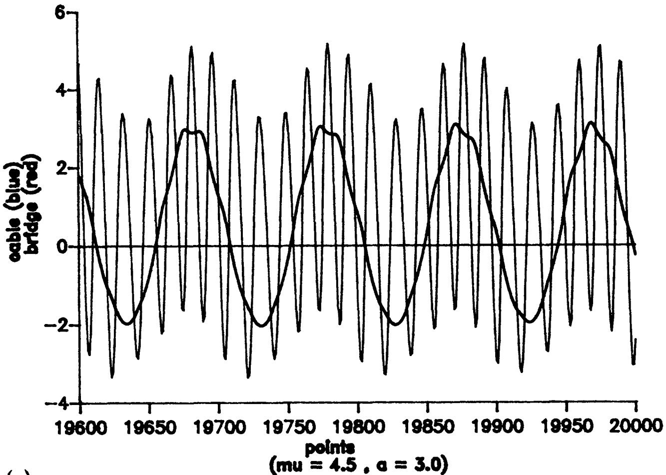


FIG. 9. *The different types of oscillation as the magnitude of the forcing term increases, in the large time solution of the bridge-cable one-dimensional system.*

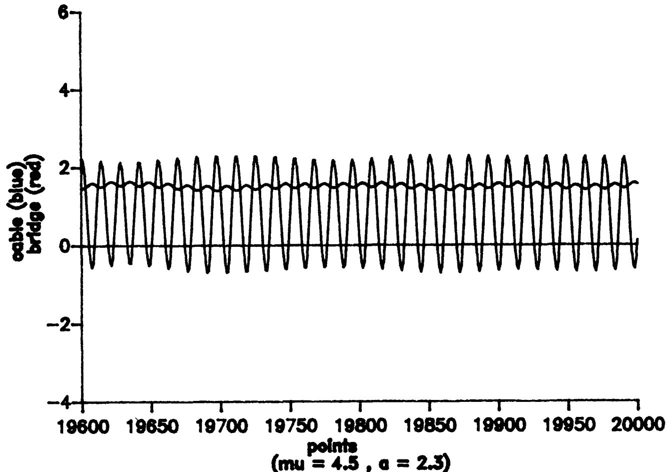
3.5. Some tentative ideas on torsional oscillation. This is the motion of the Tacoma Narrows Bridge which most people recall from the films of the event, and indeed the one which was responsible for the ultimate destruction of the bridge. We recall [5] that the bridge had been oscillating violently in a *one-dimensional* fashion during an unusually powerful storm. Then, the motion changed rapidly from the one-dimensional to a twisting torsional type, which persisted for approximately forty minutes, after which segments of the bridge began to fall into the river.

The film, and indeed contemporary reports, are clear that the cables and stays supporting the roadbed were alternatively loosening and tightening, thus creating a nonlinear effect.

It is our hope to explain this sudden transition, but this remains a long way off. However, we began by trying to understand the dynamics of a far simpler model. It is our belief that if the simplest model has some of the above described behavior, then so will the more accurate model of the bridge, when eventually formulated in terms of a long narrow vibrating plate, coupled at its side boundary with the motions of the



(c)



(d)

FIG. 9. *Continued.*

cables in the usual nonlinear way.

Thus, we consider a rigid beam of length $2l$, suspended at its ends from two nonlinear cables, of spring constant k (recall that this means that they resist *expansion* by k-times the distance, but do not resist compression). This is illustrated in Fig. 10.

The bar has weight 10 uniformly distributed along its length, and will normally be in equilibrium when the cables are in tension. The motion of the bar will be described by the two variables y, which measures the vertical distance of the center of gravity from the position of the unflexed cables, and θ, which measures the angle of the bar from the horizontal.

A solution which has a large y component and a small θ component would be primarily a vertical motion, whereas a solution with a small y component and a large θ component would be primarily a torsional motion.

We will also add two factors. There will be a small viscous damping term, which would naturally be present, and a small linear restoring term, which we added to prevent rotational motions. Clearly, the longitudinal strength of the bridge would provide some such force.

We now have a simple nonlinear mechanical system, and we can ask about large

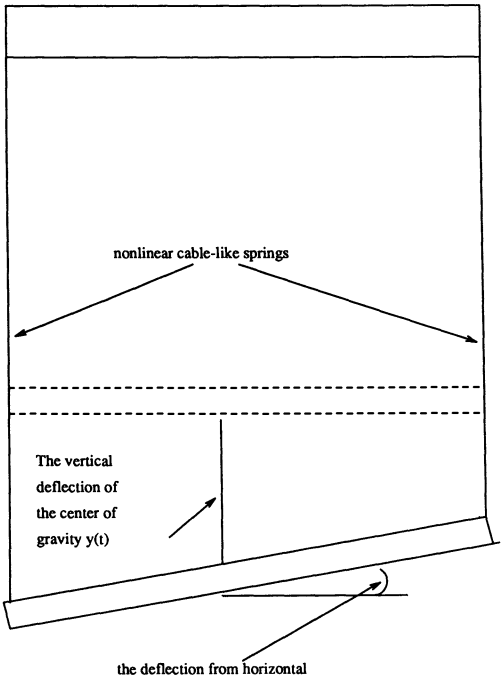


FIG. 10. *An inflexible rod, supported by two nonlinear springs and pulled downward by its own weight gives rise to a system such as* (28).

amplitude periodic solutions for various imposed forcing terms, which might be of aerodynamical or mechanical origin.

An example of such a system is the following:

$$(28)\quad \begin{aligned}\theta'' &= -.01\theta' - 3\theta + (3k/l)\cos(\theta)[(y - a\sin\theta)^+ - (y + a\sin\theta)^+] + a_1 \sin\mu t,\\ y'' &= -0.01y' - 3y - k[(y - a\sin\theta)^+ + (y + a\sin\theta)^+] + 20 + a_2 \sin\mu t.\end{aligned}$$

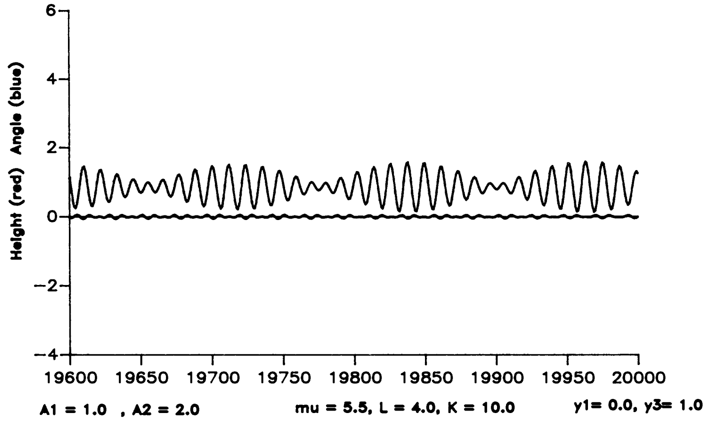
Usually the spring constant k will be taken as 10 for the rest of this section. This gives an equilibrium of $y \equiv 20/23$ and $\theta \equiv 0$.

Note some features of this system. If the motion is small, then y will be close to 1 and θ will be close to zero. Thus, the term $(y - a\sin\theta)^+$ will be the linear term $(y - a\sin\theta)$, as will $(y + a\sin\theta)^+$. Thus, we get the linear spring equation for y and the forced pendulum equation for θ. Interest in such equations has been renewed recently, starting with [12].

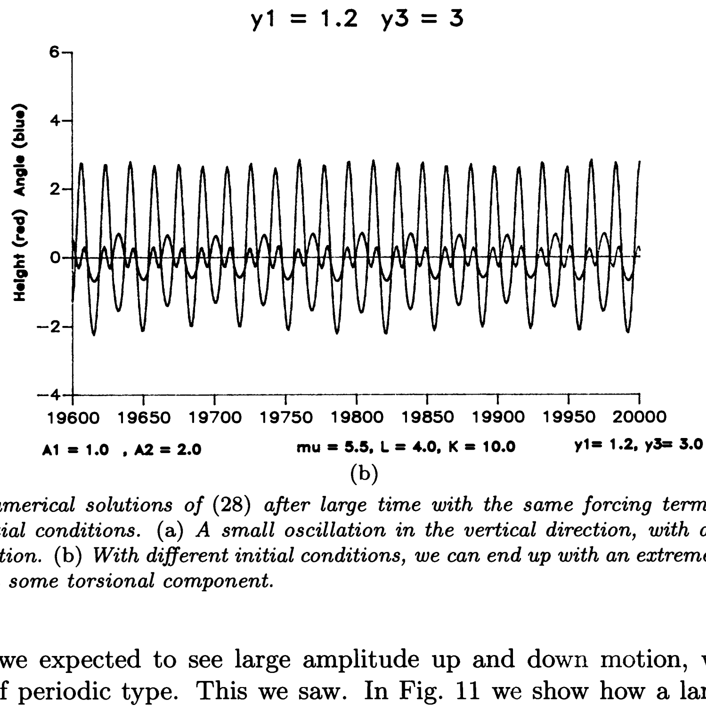
If we finally approximate $\sin\theta$ by θ, then we get a pair of uncoupled linear equations. Thus, we expect that the linear model will give good accuracy for the small oscillation case.

Our question is different. We ask whether (10) has multiple periodic solutions for various forcing terms, and if it does, what do they look like? In the case where θ' and y' are absent, the problem is variational and the method of [13] can be used to find multiple solutions. See also [43].

We are not in a position to give a complete description of the numerical results which we have obtained on this dynamical system, but we include two results which are representative, and indicate that this system does indeed exhibit the behavior that we are attempting to model.



(a)



(b)

FIG. 11. *Numerical solutions of* (28) *after large time with the same forcing term, although with different initial conditions.* (a) *A small oscillation in the vertical direction, with an imperceptible torsional motion.* (b) *With different initial conditions, we can end up with an extremely large vertical motion, with some torsional component.*

First, we expected to see large amplitude up and down motion, with multiple solutions of periodic type. This we saw. In Fig. 11 we show how a large and small vertical periodic motion, with the same forcing term can coexist, with the eventual result depending on the initial values, whereas, in Fig. 12, we show how, again with the same forcing term, we can have *torsional* or vertical motion, depending on the initial values.

This, we feel, is the likely explanation of the destruction of the Tacoma Narrows Bridge. An impact, due either to an unusually strong gust of wind, or to a minor structural failure, provided sufficient energy to send the bridge from one-dimensional to torsional orbits. We expect that this phenomenon will be shown to exist for reasonable models of the bridge, not just for the suspended rod.

3.6. Some last comments and some self-criticism. All of the models which we have discussed are painfully inadequate. For example, if one were hoping to model a real bridge, it would be impossible to ignore the coupling that exists between the main span and the side-spans.

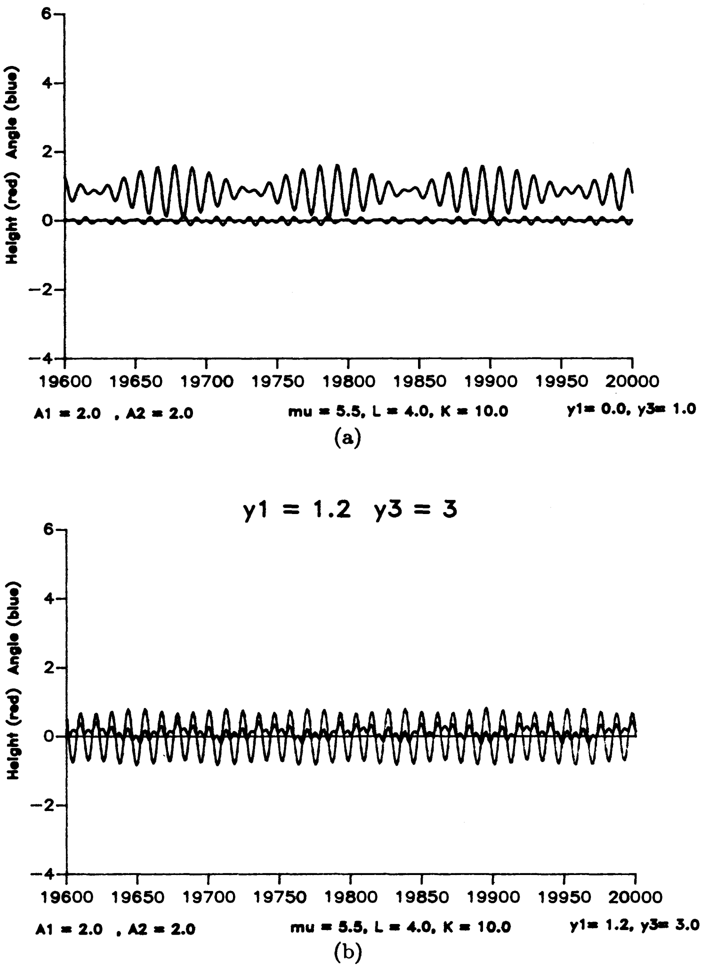


FIG. 12. *The real explanation of the Tacoma Narrows Bridge failure? With the same forcing term, the rod can go into large vertical motion with only a small torsional component* (a), *or a large torsional oscillation with a small high-frequency shaking in the vertical direction* (b).

This occurs in at least two ways. First, as either span is deflected, this results in motion of the cables and towers, thereby transmitting, through the stays, a forcing term on the other spans. Also, the boundary conditions for the main span are coupled with those of the side-span, rather than having each span act as if hinged at both ends. A complete model would also take into account the motions of the towers. These are known to exhibit large amplitude oscillatory behavior [42].

We have used only the simplest form of the beam equation. We are aware that there are more complicated versions, but we feel that if the simplest form has extremely complicated behavior, then the more complicated forms will surely have at least as complicated behavior.

We could also question whether our numerical analysis represents true solutions of the bridge equation. To some extent, we feel that this is not so important, since either the continuous or discrete versions are probably equally good (or bad) models for the bridge.

However, this does suggest an intriguing new design for a flexible suspension bridge. The problem, according to our theory, lies in the large asymmetry between

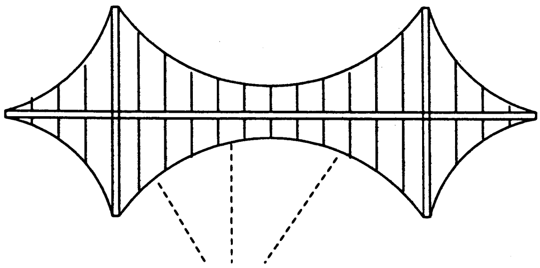


FIG. 13. *The Lazer–McKenna light flexible long-span suspension bridge. It would be equipped with cables above and below the roadbed, so as to make the structure more linear and hence less prone to large-scale oscillation. There is some evidence, numerical and historical, that this might work* (*see* §3.6).

the downward and the upward direction. A bridge less prone to oscillation would be created if the interval of asymmetry was made smaller, by having approximately equal restoring forces in the upward and downward direction. One way to do this would be to have two cable systems, one above the roadbed and one below, thereby making the bridge more symmetric and less nonlinear. See Fig. 13.

On a lighter note, it may strike the reader as ironic that the main conclusion of all this nonlinear analysis is that the engineers should attempt to *linearize* the bridge.

Surprisingly, there is some engineering evidence that this might work, and it comes from two sources. First there is the case of the first Tacoma Narrows Bridge.[4] Apparently, one of the measures taken to try to stabilize the bridge was to sink one large concrete block on the riverbed beneath each of the side-spans and tie the blocks to the side-spans of the bridge with cables.

This worked in the sense that it had the effect of quieting the motion in the side-spans, but did nothing about the motions of the main span. However, these cables were not as strong as those of the superstructure of the bridge.

A second example of the efficacy of tie-down cables is given by the history of a suspension bridge between Lewiston, New York, and Niagara, Canada, spanning the Niagara River [7]. This bridge was at the time the longest bridge in the world (1043 feet). It was partially stabilized against motions by guy cables, extending from the roadbed downward to the side of the gorge over which the structure passed. Built in 1850, the bridge lasted without incident for fourteen years until the winter of 1864–65.

At that time, due to the formation of a large ice-jam upriver from the structure, the tie-down cables were removed. It was apparently feared that when the jam broke in the spring, the ice would fall on the guys, and damage it. Within a short period of time, the bridge was destroyed by a heavy wind.

An engineer, when discussing some of these results with one of us, translated our results into simple engineering terms when he remarked, "What you are saying is that the bridge was held up enough, but not held down enough."

[4] Again we quote from the *New York Times* of November 9, 1940. "Before the collapse, temporary ties (cables) had checked the vertical motion with noticeably good effect. Unfortunately, the studies had not progressed far enough for a permanent stabilizer to be designed. *That they will be pushed to a conclusion there can be no doubt. The evolution of the suspension bridge depends on it.* (emphasis added)"

Engineers are also experimenting with ways of holding the bridge down. In [26], it is suggested that a bridge be equipped with two pipelines, running the length of the bridge on each side of the road, which in normal circumstances would remain empty. However, once strong winds develop, they would be filled with water, thus having the effect of adding additional mass to the bridge. Our theory suggests that this could be extremely dangerous, since increasing the mass could have the effect of increasing the magnitude of the oscillation, once it occurs.

One final remark: there are several long span flexible suspension bridges in earthquake-prone parts of the world, such as California and Japan. An earthquake is precisely the large amplitude forcing term which should cause these large amplitude oscillations. If the civil engineering community decides not to implement our suggestions, then maybe they could at least agree to a continuous monitoring of the bridges by video-cameras.[5] In this way, when "the big one" comes to California, the bridges, in the short interval before their final collapse, will provide confirmation of our theory, and perhaps, valuable data for its continued refinement.

4. An unexpected connection with naval architecture. The purpose of this section is to begin a study of the effect of nonlinear oscillations in floating beams [51], [9]. In naval architecture, a ship is frequently modelled as a floating beam. However, the nonlinear effects that occur when the ship is partly out of the water (known in the literature as "bottom slamming") or partly submerged (known as wetting) have not been the subject of much study. Thus, we would expect the predictions of linear theory to be quite accurate for small oscillations but not necessarily for large ones.

Although no catastrophic failures of ships have been directly attributed to large-scale oscillations of ships, the authors have found at least one case which we feel can only be explained by the presence of large-scale flexings, the destruction of the U.S.S. *Orion*.

In [51], on December 2, 1925, this ship was steaming out of the Chesapeake Capes, very nearly head on into a winds of force 9 on the Beaufort scale, approximately 50 m.p.h., which had been blowing onshore for at least 24 hours. The draft of the ship was 29.8 feet forward and 31.2 feet aft. The charted depth in the area being crossed by the ship was at least 35 feet. The waves were reported by the ship personnel to be 150 feet long and 10 feet high.

Within 10 minutes of the time that the upsea course was taken,

[5] This was written some time before the California earthquake of October 1989. According to witnesses who were on the Golden Gate Bridge at the time, the bridge did indeed go immediately into the nonlinear regime, with the stays connecting the roadbed to the cables alternately loosening and tightening "like spaghetti." The bridge oscillated for about one minute, about four times as long as the actual earthquake. However, possibly due to the angle of incidence of the earthquake waves, it did not develop significant torsional modes. Since an earthquake of up to ten times this magnitude and duration can reasonably be expected, this experience reinforces the need for understanding nonlinear behavior in suspension bridges.

On a more conjectural note, the collapse of the double-decker interstate I-880 was peculiar in the fact that although the upper deck collapsed immediately, most of the lower deck remained intact. One possible explanation for this is that the upper deck was supported only from below, rather like the nonlinear beams of §3.2. On the other hand, the lower deck was held in place from below by the supports, and from above by the weight of the upper deck. Thus, one might be led to reason that the upper deck, supported as it was by the poorly reinforced concrete columns (which resist compression but not expansion), would be more vulnerable to travelling wave behavior.

To judge from eyewitness accounts from the *New York Times*, October 19, 1989, this may have happened. "I looked in my rearview mirror and I saw the highway coming at me like a wave. The freeway started to go up and down like waves on the ocean," was an account of a survivor on the upper roadway.

> the ship began to experience "excessive vibration".... Although the record is not clear, the ship was undoubtedly pounding its forefoot on the bottom at this time. Following every impact, the hull would vibrate for a few seconds, probably until just before the next impact.
>
> *Although slowed to half speed,* the "vibrations" continued to from 30 to 40 minutes.... The ship was turned around in the sea and taken back to port, where its bottom was found to be pushed upwards and fractured. The inner bottom, for a distance of from about 25 to 130 feet abaft the stem, was found to be buckled and distorted.
>
> Despite the reported small size of the waves relative to the ship, the bow apparently pounded on the sea bed intermittently for about 20 minutes. The final damage was so severe that the ship was unloaded, decommissioned at once, and never again put back into military service. (emphasis added)

Given that this represents a forced oscillation problem, it seems clear that flexing of the ship must have played a role in this curious oscillation that seemed to favor one end of the ship. It would be difficult to explain what sort of forcing term could have created this effect in the linear theory.

The other thing that seems clear from the admittedly sketchy accounts that are available to us is that the *frequency* of the forcing term cannot have been as important as would have been expected if we were seeking to explain this phenomenon solely using the frequency of the forcing term.

In this section, we consider a nonlinear model, which takes into account that the ship may be partly lifting out of the water or partly submerged, but not both. We show that this causes large amplitude oscillations that would not be predicted by the linear theory. These oscillations occur in a wide range of frequencies, and often several different periodic oscillations can coexist for the same forcing term. In this situation, whether the ship goes into large or small oscillations depends on the initial conditions. *Furthermore, numerical results indicate that the magnitude of the oscillations increases as the frequency of the forcing term is decreased.* It is also a striking feature of this model that asymmetric solutions, with oscillations favoring one end, are predicted in the presence of almost symmetric forcing data.

The authors also feel that this may be an explanation for catastrophic failures such as the wreck of the S.S. *Edmund Fitzgerald*, about which we shall say more in §4.4.

In the first section, we derive the equation for the floating beam, we then state a theorem about large amplitude periodic flexings for this equation, and we briefly describe the results of some numerical calculations.

4.1. The floating beam: Almost a suspension bridge. Consider a rectangular block of cross section A floating in water. Assume that the difference in density between the block and water is ρ. Let $U(t)$ denote the depth of the bottom of the block as it floats. Assume that the block is floating high in the water, so that it may lift out of the water but is never submerged. Then the force pushing the block up in the water is given by $\rho U(t)A$ if U is positive and zero if U is negative. Thus, if there is an external forcing term, as well as the force of gravity, the equation satisfied by the block is given by

$$\frac{\partial^2 U}{\partial t^2} = c - bU^+ + f(t) \tag{29}$$

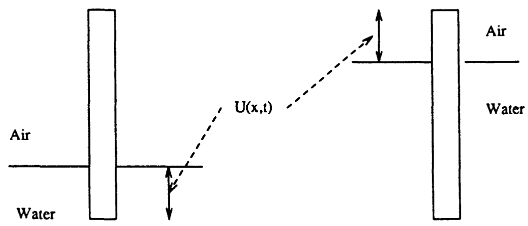


FIG. 14. *A cross-section of the floating beam at position x, in (left) the lightly loaded case and (right) the heavily loaded case.*

for suitable constants b and c.

Now consider the case where the block is almost completely submerged and only the distance measured upwards, $U(t)$, is above the water. In this case, two forces will act on the block. There will again be a force due to flotation of $LA\rho$ if the block is completely submerged or $(L-U(t))\rho$ if the top of the block is a distance U out of the water. This force will act in the positive U direction. In the negative direction, there will again be a force W, due to gravity. Again, the equation satisfied by the block will be

$$m\frac{\partial^2 U}{\partial t^2} = LA\rho - W - A\rho U^+ + f(t). \tag{30}$$

In either case, the equation satisfied is of the form

$$m\frac{\partial^2 U}{\partial t^2} + BU^+ = C + f(t) \tag{31}$$

for suitable constants B and C. See Fig. 14.

If we consider the case, not of a floating block, but a floating beam of length L, it is clear that the equation will be of the form

$$U_{tt} + U_{xxxx} + \delta U_t + aU^+ = c + f(x,t), \tag{32}$$

where now $U = U(x,t)$, where $0 \le x \le L$ and where the ends of the beam satisfy free-end boundary conditions, i.e.,

$$U_{xx}(0,t) = U_{xx}(L,t) = U_{xxx}(0,t) = U_{xxx}(L,t) = 0.$$

The constant a is a measure of the cross section of the beam, and the constant δ represents the viscous damping in the beam.

It is a surprising fact that the equation of the suspension bridge and the floating beam differ only in the boundary conditions.

Here, we examine periodic solutions of this equation, subject to the free-end boundary conditions. In order to approach this problem, we must first gather some information on the operator $LU = U_{tt} + U_{xxxx}$ with these boundary conditions. First, we consider the case where the interval is $(0,\pi)$.

In turn, to do this, we must first understand the ordinary differential operator

$$\mathcal{L}y = y^{(iv)}, \quad y^{(2)} = y^{(3)} = 0, \quad x = 0, \pi \tag{33}$$

with the additional symmetry condition that U is symmetric about $\pi/2$. This operator is a self-adjoint operator with an infinite sequence of eigenvalues λ_i, and their associated eigenvectors ϕ_i.

The functions ϕ_i are given by normalizing multiples of

$$\hat{\phi}_i = a_i \cos(\nu_i(x - \pi/2)) + b_i \cosh(\nu_i(x - \pi/2)), \tag{34}$$

where the ν_i are the successive zeros of $\tanh(\nu\pi/2) + \tan(\nu\pi/2)$ and $a_i = \cosh(\nu_i\pi/2)$ and $b_i = \cos(\nu_i\pi/2)$. The corresponding λ are given by $\lambda_i = \nu_i^4, i > 0$. Of course, $\lambda_0 = 0$, with the corresponding $\phi_0 = 1$. (Recall that the eigenfunction $(x - \pi/2)$ is ruled out by the symmetry condition.) The functions $\{\phi_i, i \geq 0\}$ are an orthonormal basis for the Hilbert space $\mathcal{H} = L^2(0,\pi) \cap \{\text{functions even about } \pi/2\}$.

If we are interested in periodic solutions of periodic $2\pi/k$ of the partial differential equation, we will look in the space $H_k = \mathcal{H}_x \oplus L_t^2(0, 2\pi/k) \cap \{\text{functions even in } t\}$. By our earlier remarks, we have an orthonormal basis of this space given by $\Phi_{m,n} = \phi_m(x)\cos knt, m, n \geq 0$ with associated eigenvalues $\Lambda_{m,n}$. The unbounded operator L is a self-adjoint operator on the space H_k and the functions $\Phi_{m,n}(x,t) = \phi_m(x)\cos knt, m, n \geq 0$ are eigenfunctions of L with eigenvalues $\Lambda_{m,n} = -k^2n^2$. Note that an easy calculation gives that the ν_i rapidly approach $2m - 1/2, m = 1, 2, \cdots$, and thus the $\Lambda_{m,n}$ rapidly approach $(2(m - 1/4))^4 - k^2n^2$.

Some comments about the model are in order at this stage. If we assume that there is no forcing term f then (31) has a unique steady state solution, $u(x,t) \equiv c/a$ and this is globally attracting. If f is small and the solution u of (4) is of the same order of magnitude as f (i.e., we are not in a situation of linear resonance), then to solve (4) we need only solve the linear equation

$$U_{tt} + U_{xxxx} + \delta U_t + aU = c + f(x,t). \tag{35}$$

Therefore, we can find periodic solutions of (31) simply by finding the well-understood solutions of the linear equation. These solutions will be (away from resonance) of the form $c/a + \hat{u}$, where $\hat{u}$ is of the same order of magnitude as the forcing term $f(x,t)$. This is what we would call the intuitively obvious solution: a small forcing term results in a small perturbation about equilibrium.

However, if the forcing is small, then we can ask two related nonlinear questions: first, is this the only periodic motion and, if other periodic solutions exist, are they stable?

As the reader should suspect by now, other solutions exist, and they appear to be stable.

4.2. An abstract theorem. Let Ω be a bounded domain in $\Re^n$, let H denote a real Hilbert space which is a closed subspace of $L^2(\Omega)$ where we denote the usual L^2-inner product by $(\ ,\ )$. Let $L : D(L) \subseteq H \to H$ be a self-adjoint operator. We shall discuss an abstract operator equation

$$Lu + bu^+ = c\psi_0, \tag{36}$$

which we shall later relate to the differential equation.

We make the following assumptions:

Assumption 1.

$$\text{Dim Ker}(L) = 1 \quad \text{and} \quad \text{Kernel } (L) = \{s\psi_0 \mid s \in \Re\}. \tag{37}$$

Assumption 2. $b > 0, c > 0$ and there exists $\psi_1 \in H$ with ψ_1 not $\equiv 0$ such that $L\psi_1 = -\alpha\psi_1$, where $0 < \alpha < b$.

Assumption 3. There exists a number $d > 0$ such that if $| s | \leq d$, then for all $x \in \Omega$

$$\psi_0 + s\psi_1 > 0.$$

Assumption 4. There exist numbers r_1 and r_2 with

$$r_1 < -b < 0 < r_2 \tag{38}$$

such that if V is the two-dimensional subspace of H spanned by ψ_0 and ψ_1, L_1 is the restriction of L to the invariant subspace $V^\perp$ of H and σ_1 denotes the spectrum of L_1, then

$$\sigma_1 \subset (-\infty, r_1) \cup (r_2, \infty).$$

Using Assumptions 1 and 3, we find by inspection the obvious solution $u = c\psi_0/b$ of (35). The following result gives us two more solutions which are less obvious.

THEOREM 4.1. *Under Assumptions 1–4, there exist solutions u_1 and u_2 of* (35) *such that $(u_1, \psi_1) > 0$, $(u_2, \psi_1) < 0$.*

Proof. See [36].

PROBLEM 12. *Can the restriction $r_1 < -b$, which is necessary to the method of proof, but not natural to the result, be removed? Also, since the restriction $r_1 < -b$ is a limitation of the method of proof, we might expect more than four solutions when $r_1 > -b$.*

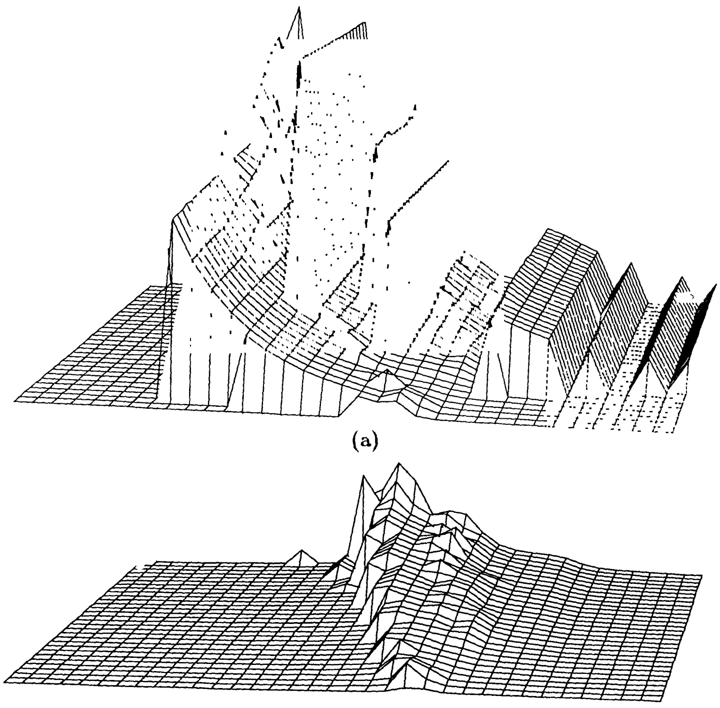
For example, we take the beam to be of length π, then the space V will be spanned by the constants (which are the ψ_0) and the function ψ_1 will be given by $\phi_1(x)\cos(3t)$, with corresponding eigenvalue -3.86122036. The next negative eigenvalue corresponds to $\cos(3t)$, and is 9.0. Here we define L to be the operator $Lu = u_{tt} + u_{xxxx}$ (with free-end boundary conditions) defined on the space $H_3 =$ closure span of $\{\phi_m(x)\cos 3nt, m \geq 1, n \geq 0\}$. Thus, we are looking for periodic solutions of (31) which are even in t, even about the midpoint of the beam, and periodic of period 2π. Therefore, what the main theorem of the previous section shows is that if $3.86122036 < b < 9.0$, (31) has at least two large amplitude oscillatory solutions with a nonzero component of period $2\pi/3$. It is a standard result that if $0 < b < 3.86122036$, then (31) has at most one solution, the obvious $u \equiv c/b$. Thus, when $-b$ crosses the eigenvalue -3.86122036, we create additional solutions for a wide range of values of b.

4.3. Some numerical results. We discuss numerical solutions of (31). To study the equation numerically, we will search for periodic solutions in a fairly naive way, by solving the initial value problem for various initial values and allowing the solution to run for large time. If there is a unique periodic solution, then it is reasonable to hope that after large time, the solution will have converged to it. We emphasize that we have little interest in the intermediate values of the solution to the initial value problem, only the eventual long-time behavior.

We solve the equation

$$U_{tt} + U_{xxxx} + 0.01U_t + 18U^+ = 10 + f(x,t) \tag{39}$$

under a variety of assumptions on the interval, on f, and on the initial values. Our method of solution is to use the finite-difference method, implicit in the linear part and



(a)

(b)

FIG. 15. *The amplitude-frequency response after large time, for solving the initial value problem with large* (a) *or small* (b) *initial deflections, for* (39) *with free-end boundary conditions.*

explicit in the nonlinear part $18U^+$ and in the forcing term. After considerable experimentation it was found that this method gave best agreement with analytic results (where available) and also gave the best results when meshsizes were considerably reduced as a control on accuracy.

Our forcing term was generally of the form $(\lambda \sin \mu t)F(x)$, which is the form of a standing wave. Thus, for fixed $F(x)$, we would continuously vary λ and μ over a wide range of values and study the long-term solutions.

A natural choice of $F(x)$ would be $\cos(2\pi x/L)$, which would induce a two-node flexing with maximum bending about the midpoint, as long as the oscillation remained in the linear range. The picture is the same as for the bridge, and is fully detailed in [36].

Figure 15 shows the resulting magnitude of large and small initial data, after the initial value problem has been solved for large time. Large amplitude nonlinear oscillation is present over a wide range of λ and μ. The forcing term $f(x,t)$ is taken to be $\lambda \cos(2\pi x/L)\sin(\mu t)$. The amplitude of the large-time solution is shown as a function of λ and μ. The parameter μ varies from left to right, in increments of 0.2, from 0.2 to 7.0. For small λ, the two solutions appear to agree, except below linear resonance. For λ greater than 0.2, convergence to different amplitude solutions is indicated for large initial values. For μ above resonance, the large amplitude solutions are actually motions of frequency half that of the forcing term.

Perhaps the most startling thing about some of these oscillations is the form they sometimes take, as shown in Fig. 16, with an asymmetry in the solutions, no longer being a flexing about the midpoint, but showing a distinct preference for one end. Over many different experiments, we found this type of solution occurring, with

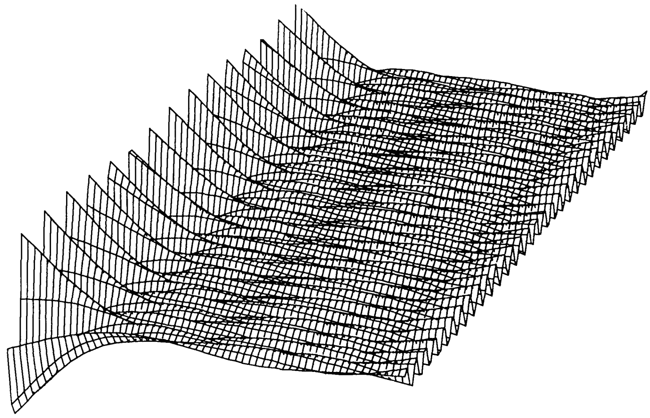


FIG. 16. *A finite difference solution to* (38), *with the forcing term* $f(x,t)$ *taken to be* $\lambda\cos(3\pi x/L)\cos(\mu t)$. *With large initial conditions, the finite difference solution converges to a large scale flexing which favors one end. This should be compared to the account of the destruction of the* Orion.

either end emerging as the preferred end. It seems as if a form of symmetry-breaking occurs, with the symmetric periodic solution presumably becoming unstable.

PROBLEM 13. *Show that unsymmetric solutions exist for the unforced problem. Investigate this as an occurrence of symmetry-breaking.*

4.4. Some remarks on the naval architecture literature. In this section, we review three separate cases from the literature which do seem to confirm some of our findings. We summarize our principle findings as follows:

1. Decreasing the frequency away from resonance may have the effect of increasing the amplitude of the oscillation.
2. Large amplitude and small amplitude periodic flexings may coexist for the same forcing term; an unusual combination of conditions may result in changing from one to the other.
3. Over a wide range of frequency, as soon as the nonlinear effects of deck-wetting or lifting out of the water occur, oscillations which favor one end may occur.
4. Large symmetric periodic flexings may occur in which the bending moment is quite small at the midpoint but becomes large at two points some distance from the midpoint.

In [2], one situation similar to our first principle was observed. In a sea state Beaufort 7, on an ore-carrier being observed for just this purpose, the ship was encountering conditions which induced slamming at a certain rate. When the speed and direction of the ship was changed to almost half the rate of slamming, the measured whipping stress in the ship was almost doubled. The author comments, "This odd behavior of the whipping stress ought be mentioned." We emphasize that this is not the exact situation of our paper, since the ship was apparently partially lifting out of the water as well as partially submerging.

The case of the destruction of the *Orion*, which we mentioned at the beginning of this section, provides further evidence for our conclusions 1 and 2. The standard sea-going practice of reducing the frequency by slowing down was put into effect. Indeed the ship immediately halved its speed. This did not stop the oscillation, and indeed, it may have made it worse. Furthermore, the oscillation was clearly favoring one end,

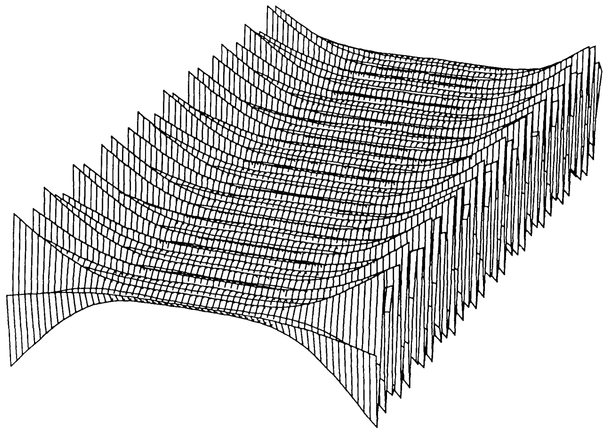


FIG. 17. *The nonlinear flexing induced by a forcing term of the type* $\lambda \cos(2\pi x/L)\cos(\mu t)$. *In the nonlinear range, the greatest bending moment occurs at two points approximately equidistant from the mid-point. The* Edmund Fitzgerald *broke at such points.*

since only the front of the ship was hammering on the sea bed.

The case of the loss of the *Edmund Fitzgerald* is at least suggestive. This ship, in heavy seas on Lake Superior, vanished with large loss of life. In [59], it was concluded that

> the proximate cause of the loss of the Edmund Fitzgerald cannot be determined ... the end was so rapid and catastrophic that there was no time to warn the crew, ..., or even to make a distress call.

The report goes on to make conjectures that relate to the ship being slowly waterlogged without the crew being aware of it although, apparently, the pumps were working.

> Finally, as the storm reached its peak intensity, so much freeboard was lost that the bow pitched down, and dove into a wall of water, and the vessel was unable to recover. Within a matter of seconds, the cargo rushed forward, the bow plowed into the bottom of the lake, and the midship's structure disintegrated.

We believe that it is at least as likely (rather than the "wall-of-water" theory) that the ship went into a large-scale flexing motion which was mistaken by the crew for normal free rigid motion in very heavy seas. (The visibility was extremely low.) This would also account for one of the most puzzling aspects of the case, namely why the ship was not broken at its midpoint but at two points approximately 80 feet from the midpoint. This is precisely what would happen if the ship was oscillating in the mode shown in Figs. 16 and 17.

5. Some concluding remarks. In this review, we have seen an area go from one of basically pure interest to one of significant applied and engineering interest. We start with some abstract existence results. These results, in turn, are confirmed by the numerical calculations, which suggest both new theorems, and new applications of engineering interest.

Finally, these results make suggestions about how large structures should be constructed, or how ships should be handled at sea.

Our belief is that there will be many other applications of this type of analysis. For example, the semilinear beam equation that we have used for bridges and ships also has occurred in the civil engineering literature, when the deflections of a railroad track were studied [37].

We also expect more progress on the application of current research in Hamiltonian systems to system problems of the sort which arose in the bridge section, where we have barely scratched the surface.

Finally, we expect that in the future some of these results will have application in the study of electric circuits, where most of the present work treats capacitors as linear, even though they are inherently nonlinear.

Acknowledgments. Much of the numerical investigation which resulted in Figs 8, 9, 11, and 12 was done by Steve Martino, Carolyn Lebenski, and Joseph Tokarski as part of their research experience as undergraduates at the University of Connecticut. We would like to thank Paul Davis, who produced the initial impetus for this paper.

Note added in proof. It has been pointed out to us that we have not cited a recent paper in the engineering literature, "Suspension bridge vibration: continuum formulation" by Ahmed M. Abdel-Ghaffer in the Journal of Engineering Mechanics division, Proc. of A.S.C.E., 108 (1982), pp. 1215–1232. This paper derives nonlinear equations for suspension bridge vibrations.

There are problems. Before deriving the equations, the author assumes that $\sin\theta = \theta$, (where θ is the angle of torsion), assumes that the main cables satisfy the linear (small amplitude) vibrating string equation, and assumes that the hanger cables behave like rigid incompressible rods. (In the Tacoma Narrows Bridge, they were seen alternately to loosen and tighten.) These assumptions are not explicitly mentioned in the text, but are implicit in the derivation.

The author then derives "the coupled equation of motion in their most general and nonlinear form." Initial values or conditions are never mentioned in the paper. Boundary values are never specified in the variational formulation. No nonlinear behavior of the solutions is investigated.

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