A VERIFICATION OF LANCHESTER'S LAW†

J. H. ENGEL

Operations Evaluation Group

Massachusetts Institute of Technology††

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The validity of Lanchester's equations is demonstrated in an actual combat situation where U.S. forces captured the island of Iwo Jima. The equations tested are $dM/dt=P(t)-AN$, and $dN/dt=-BM$, with

t the time elapsed since the beginning of the engagement, measured in days;

$M(t)$, and $N(t)$ the number of effective friendly and enemy troops, respectively;

A and B the friendly and enemy combat loss rates, respectively, per opposing combatant; and

$P(t)$ the rate friendly troops enter combat.

Information required for the verification is the number of friendly troops put ashore each day and the number of friendly casualties for each day of the engagement, knowledge that enemy troops are not reinforced or withdrawn, the number of enemy troops at the beginning and end of the engagement, and the time T at the end of the engagement. If the number of enemy troops at the beginning of the engagement is not known, but the time, T, when the enemy is *destroyed* is known, it is still possible to determine whether Lanchester's equations are valid. In the analysis of the capture of Iwo Jima, the equations were found to be applicable. The value of such analyses increases when repeated often enough to permit general conclusions to be drawn.

IN 1916, F. W. Lanchester pointed out the importance of the concentration of troops in modern combat.[1] He proposed differential equations from which the expected results of an engagement could be obtained. His conclusion that the strength of a combat force is proportional to the square of the number of combatants entering an engagement has become known as 'Lanchester's square law.' For a discussion of Lanchester's equations[2] (including a probability analysis) and a detailed mathematical treatment,[3] the reader is referred to the literature.

It is the purpose of this paper to show how to verify the applicability of certain generalized Lanchester equations to a simple combat situation where there is a strong *a priori* reason for believing that the equations are valid.

ANALYSIS

The combat situation that will be discussed is one in which our own forces surrounded and captured Iwo Jima. During the engagement, enemy

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†† Navy Department, Washington 25, D.C.

troops were neither withdrawn nor reinforced. At the termination of the engagement all enemy troops had been destroyed. During the first few days of the engagement our forces landed varying numbers of troops. Operational losses (those caused by accidents, weather, etc., rather than enemy activity) were negligible and could be ignored.

Under these circumstances, the Lanchester-type equations that appear to provide a reasonable description of the situation are:†

$$dM/dt = P(t) - AN, \tag{1a}$$

$$dN/dt = -BM, \tag{1b}$$

with:

t the time elapsed since the beginning of the engagement measured in days;

$M(t)$, $N(t)$ the number of effective friendly and enemy troops, respectively, taking part in the engagement at any time, t;

A, B the friendly and enemy combat loss rates, respectively, per opposing combatant;

$P(t)$ the rate at which friendly troops enter combat.

These equations are based upon the assumption that *the casualty-producing rate of an entire force is equal to the number of troops in the force multiplied by the casualty-producing rate of the average combatant in the force.*

These equations may be solved by quadratures for any integrable function, $P(t)$, yielding $M(t)$ and $N(t)$ as functions of $P(t)$, A and B, as follows:

$$\begin{aligned} M = M(0)\cosh\sqrt{AB}\,t - \sqrt{A/B}N(0)\sinh\sqrt{AB}\,t \\ + \int_0^t \cosh\sqrt{AB}(t-s)\ P(s)ds, \end{aligned} \tag{2a}$$

$$\begin{aligned} N = N(0)\cosh\sqrt{AB}\,t - \sqrt{B/A}M(0)\sinh\sqrt{AB}\,t \\ - \sqrt{B/A}\int_0^t \sinh\sqrt{AB}(t-s)\ P(s)ds, \end{aligned} \tag{2b}$$

where $M(0)$, $N(0)$ are the number of friendly and enemy troops on hand at the moment the engagement starts.

Whether or not these equations can be applied to a given situation depends upon whether or not it is possible to assign values to A and B in such a manner that the observed values of M are in good agreement with the values obtained from a solution of equations (2).

† These equations are specializations of the following equations (in which P and Q are production rates, A and B combat loss rates, C and D operational loss rates):

$$dM/dt = P - AN - CM, \qquad dN/dt = Q - BM - DN.$$

An examination of the available records[4] disclosed that the following data were available for the Iwo Jima battle:

(a) the total number of friendly troops put ashore each day (no friendly troops ashore prior to the beginning of the engagement);

(b) the total number of friendly casualties each day and, separately, those killed in action;

(c) the number of enemy troops ashore at the beginning of the engagement;

(d) the time when the island was declared secure (after this time, although the battle continued, it may have done so at a different rate);

(e) the time when the engagement ended (after the island was declared secure);

(f) the number of enemy troops at the end of the engagement (zero if all were destroyed).

From these data, it was possible to determine M_t^*, the observed number of effective friendly troops on hand at the beginning of the tth day. It was also possible to determine, from the data, values of $p^*(t, t+1)$ the total number of friendly troops put ashore during time t through $t+1$. In order to characterize the function $P(t)$ for all values of t, so that equations (2) might be used, it was decided to assume that the rate that friendly troops were put ashore was constant during each day of the engagement. Then,

$$P(s) = p^*(t, t+1) \text{ for } t \leqslant s < t+1. \quad (t \text{ a non-negative integer}) \qquad (3)$$

With this simplifying assumption, and assigning $M(0)$ and $N(0)$, the observed values M_0^* and N_0^*, respectively, equations (2) may be used to derive the following equations:

$$\left.\begin{aligned} &M(0) = M_0^* = 0 \\ &M(t+1) = M(t)\cosh\sqrt{AB} - \left[N(t) - \frac{P(t)}{A}\right]\sqrt{A/B}\,\sinh\sqrt{AB}) \\ &N(0) = N_0^* \\ &N(t+1) = \left[N(t) - \frac{P(t)}{A}\cosh\sqrt{AB} - M(t)\sqrt{B/A}\cosh\sqrt{AB} + \frac{P(t)}{A}\right. \end{aligned}\right\} \qquad (4)$$

Equations (4) are stated in this form as an illustration of a solution that facilitates computation by hand with a desk computer and a minimum of entering of tables.

When values are assigned to A and B, equations (4) yield a set of values of $M(t)$ and $N(t)$. If values for A and B can be found from which values of $M(t)$ can be computed that are in close agreement with the observed values M_t^*, and with $N(0) = N_0^*$, and $N(T) = N_T^*$, then it may be asserted that the equations (4) describe the engagement being tested.

A technique for estimating the values of A and B that should apply to a specific engagement of the sort described in this paper will now be shown.

Using equation (1b) with the observed values of: $M_0^*=0$, M_t^* for $0<t\leqslant T$, N_0^*, and N_T^*, and solving for B, we obtain

$$B=(N_0^*-N_T^*)\Big/\int_0^T M_t^* dt.$$

Inasmuch as M_t^* in the example at hand was known only for integral values of t, and $M_0^*=0$, the integral in the above expression will be approximated by the summation $\sum_{t=1}^{T} M_t^*$, so that we obtain

$$B=(N_0^*-N_T^*)\Big/\sum_{t=1}^{T} M_t^*. \tag{5}$$

Other integrals that appear will be similarly replaced.

A is found as follows:

Substitute the value of B derived from equation (5) into equation (1b), integrate and solve for $\tilde{N}(t)$ (t between 0, T) obtaining:

$$\left.\begin{aligned} &\tilde{N}(0)=N_0^* \\ &\tilde{N}(t)=N_0^*-B\sum_{s=1}^{t} M_s^* \quad \text{for} \quad t>0 \end{aligned}\right\} \tag{6}$$

The values $\tilde{N}(t)$ obtained from this equation will be called the approximate theoretical values of $N(t)$, as they have been derived from observations and equation (1b) only.

Using these values of $\tilde{N}(t)$ in equation (1a) and the assumed values of $P(t)$ shown in equation (3), integrating from 0 to S, where S is some fixed time at or near the termination of the engagement, letting $\tilde{M}(S)=M_S^*$, and solving for A we obtain:

$$A=\left(\sum_{t=0}^{S} P(t)-M_S^*\right)\Big/\sum_{t=0}^{S} \tilde{N}(t). \tag{7}$$

Substituting this value of A, the assumed production rates $P(t)$, and the values $\tilde{N}(t)$ derived from equation (6) in equation (1a), and integrating, we obtain the approximate theoretical values $\tilde{M}(t)$:

$$\tilde{M}(t)=\sum_{s=0}^{t} P(s)-A\sum_{s=0}^{t}\tilde{N}(s). \tag{8}$$

These approximate theoretical values, $\tilde{M}(t)$, should be in close agreement with the observed values M_t^* in order to consider equation (1) valid.

Having obtained reasonable estimates of A and B as provided by equations (7) and (5), the verification is completed by substituting these values of A and B in equations (4), and, using the assumed values of $P(t)$ shown in equation (3), finding $M(t)$ and $N(t)$. These values are designated *exact theoretical values*, since they were derived directly from equations (1), no information concerning the day-to-day values of M_t^* being required. The choice of B assures us that the values $\tilde{N}(0)$ and $\tilde{N}(T)$ agree with N_0^* and N_T^*, respectively, and the choice of A assures us that $M(0)$ and $M(S)$ agree with M_0^* and M_S^*, respectively, but neither guarantees the resemblance of $M(t)$ to $M^*(t)$ or $N(t)$ to $N^*(t)$.

Some simpler techniques for estimating A and B can be devised but are not described in detail, as they are not as widely applicable as the technique shown here. One such technique utilizes the observation that in the battle of Iwo Jima the number of friendly troops was usually within 10 per cent of its average value, and then from the obvious result that $N(t)$ must vary linearly with t concludes most elegantly and simply that

$$A=\left(\sum_{t=0}^{T} P(t)-M_T^*\right)\bigg/ \tfrac{1}{2}[N_0^*+N_T^*][T+1],$$

for an engagement from 0 to T. While such a result is considerably simpler than the development shown in equations (6) and (7), its use must be restricted to those situations in which M_t^* has been observed to be approximately constant during the majority of an engagement. Since this is not a typical situation, this simple method will not be universally applicable, hence has not been derived in detail.

Once an estimate of the correct values of A and B has been made by equations (7) and (5) or alternative techniques, the values of $M(t)$ and $N(t)$ may be computed from equation (4) and checked against M_t^* and N_T^*.

THE BATTLE OF IWO JIMA

In the battle of Iwo Jima our forces landed 54,000 troops the first day; none on the second; 6,000 the third, none on the fourth and fifth; 13,000 on the sixth day, and none thereafter. By virtue of equation (3), therefore,

$$P(s)=\begin{cases} 54{,}000 & 0\leqslant s<1 \\ 0 & 1\leqslant s<2 \\ 6{,}000 & 2\leqslant s<3 \\ 0 & 3\leqslant s<4 \\ 0 & 4\leqslant s<5 \\ 13{,}000 & 5\leqslant s<6 \\ 0 & \text{all other} \end{cases}$$

These figures included the personnel of the specific units that were actually put ashore; they included the men in indirect support of the battle, so that the resulting determination of A and B must be interpreted as average effectiveness per man ashore. Casualty lists were available[4] from which M_t^*, the number of effective friendly troops on shore at time t could be determined. Initially, an effective troop was defined as a man on shore who was not killed, wounded, or missing in action.

The engagement ended at $T=36$, with $\sum M_t^*=2{,}037{,}000$, $N_0^*=21{,}500$, and $N_T^*=0$. B was determined from equation (5) to have the value 0.0106 enemy casualties per day per effective friendly troop.

Approximate theoretical values of $\tilde{N}(t)$ were computed, using this value of B, in equation (6).

The value of S used for computing A was 28. (The island was declared secure on the 28th day, although fighting kept up until the 36th day.) This was done because it was felt that if there were any marked variations in A and B, due to a possible change in the rate of combat, such variations would probably occur after the island was secured, when the fighting would become more sporadic. For $S=28$, it was determined that $M_S^*=52{,}735$, $\sum_0^S P_t=73{,}000$, and the evaluation of $\sum_0^S \tilde{N}(t)$ from the approximate theoretical values of $\tilde{N}(t)$ yielded 372,500. From equation (7) it then followed that A had the value of 0.0544 friendly casualties per day per effective enemy troop.

Approximate theoretical values of $\tilde{M}(t)$ were then computed from equation (8). Using equations (4), exact theoretical values of $M(t)$ and $N(t)$ were then computed. A comparison of the observed values of M_t^* with the approximate and theoretical values of $M(t)$ is shown in Fig. 1a; it appears that equations (1) are a valid description of the engagement.

The relative effectiveness of the troops in the engagement is then

$$\begin{aligned} A/B &= 0.0544/0.0106 \\ &= 5.1\ \frac{\text{friendly casualties per effective enemy troop}}{\text{enemy casualties per effective friendly troop}}. \end{aligned}$$

The nature of the enemy casualties has not been specified. There was evidence from the battlefield (corpses about equal in number at the end of the battle to the total enemy on hand at the start of the engagement, and no enemy alive at the end of the engagement) that an enemy troop continued as an effective troop until killed.

Additional computations were made in which friendly troops were counted as ineffective only if killed. The fits of the resulting approximate theoretical, exact theoretical, and observed values of M_t are shown in Fig. 1b and are as good as the fits obtained previously. The value of A obtained in this computation was 0.0113 friendly troops killed per day per effective enemy troop. The value of B was 0.0088 enemy casualties per day per effective friendly troop. The relative effectiveness computed this way became

$$0.0113/0.0088 = 1.3 \frac{\text{friendly troops killed per effective enemy troop}}{\text{enemy casualties per effective friendly troop}}.$$

The ratio of these two relative effectiveness values is

$$5.1/1.3 = 4 \frac{\text{friendly casualties}}{\text{friendly troops killed}}.$$

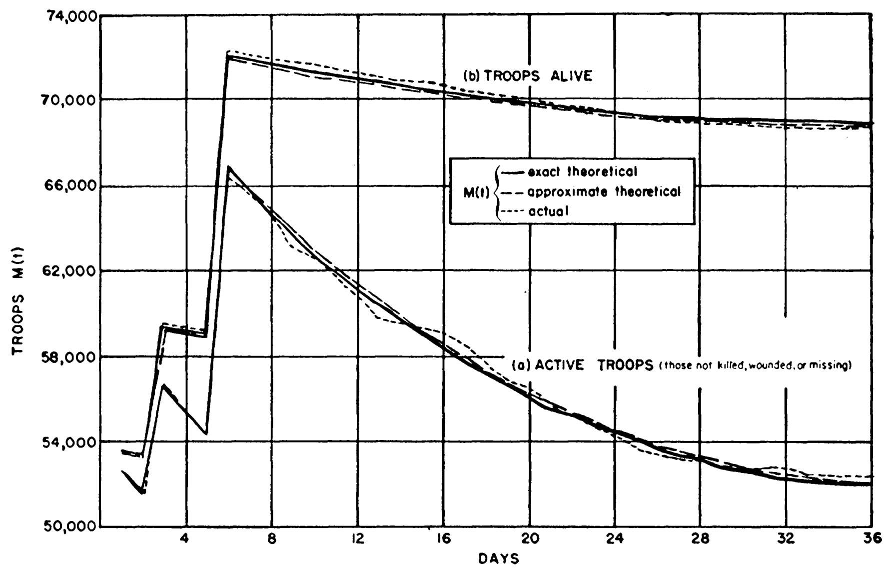

FIG. 1: $M(t)$, number of United States troops in action.

This ratio was corroborated by the statistics of 20,860 friendly casualties during the engagement, of which 4,590 were killed in action, or a ratio of 4.5. The difference between the values of 4 and 4.5 may be attributed at least in part, to rounding off errors occurring during calculations of approximate theoretical values of $\tilde{N}(t)$, as such errors affect the value of A

ADDITIONAL REMARKS

By a slight modification of the techniques described above, it is possible to verify the applicability of equations (1), even if $N_0{}^*$ is not known. If, for instance, it is known that all enemy troops were destroyed at time T, it may be stated that $N_T{}^*/N_0{}^* = 0$, even though $N_0{}^*$ is not known.

This, plus knowledge of $P(t)$ and M_t^*, is sufficient for the verification. The solutions to equations (1) may be rewritten in the following manner completely equivalent to equations (4):

$$\left.\begin{aligned} &M(0)=M_0^*=0\\ &M(t+1)=M(t)\cosh\sqrt{AB}\\ &\qquad -\left[\frac{N(t)}{N(0)}-\frac{P(t)}{AN(0)}\right]N(0)\sqrt{A/B}\sinh\sqrt{AB} \end{aligned}\right\}\text{(9a)}$$

$$\left.\begin{aligned} &N(0)/N(0)=N(0)/N_0^*=1\\ &\frac{N(t+1)}{N(0)}=\left[\frac{N(t)}{N(0)}-\frac{P(t)}{AN(0)}\right]\cosh\sqrt{AB}\\ &\qquad -\frac{M(t)}{N(0)}\sqrt{A/B}\sinh\sqrt{AB}+\frac{P(t)}{AN(0)} \end{aligned}\right\}\text{(9b)}$$

To use equations (9), estimates of $AN(0)$ and $B/N(0)$ will be sufficient; notice that

$$\sqrt{AB}=\sqrt{AN(0)\cdot B/N(0)},$$

and

$$N(0)\sqrt{A/B}=\sqrt{[AN(0)]/[B/N(0)]}.$$

These estimates of $AN(0)$ and $B/N(0)$ may be obtained in the following manner, completely equivalent to equations (5) through (7):

$$B/N(0)=B/\tilde{N}(0)=B/N_0^*=(1-N_T^*/N_0^*)\Big/\sum_{t=1}^{T}M_t^*, \qquad (10)$$

in which the numerator on the right is unity when it is known that $N_T^*=0$,

$$\left.\begin{aligned} &\tilde{N}(0)/N(0)=\tilde{N}(0)/\tilde{N}(0)=\tilde{N}(0)/N_0^*=1,\\ &\tilde{N}(t)/N(0)=\tilde{N}(t)/\tilde{N}(0)=\tilde{N}(t)/N_0^*=1-(B/N_0^*)\sum_{s=1}^{t}M_s^*,\text{ for } t>0 \end{aligned}\right\}(11)$$

taking B/N_0^* from equation (10), and

$$AN(0)=\left(\sum_{t=0}^{S}P(t)-M_S^*\right)\Big/\sum_{t=0}^{S}\tilde{N}(t)/N(0), \qquad (12)$$

taking $\tilde{N}(t)/N(0)$ from (11) above. Using $B/N(0)$ and $AN(0)$ derived from equations (10) and (12) in equations (9), $M(t)$ may be obtained and compared with observed value of M_t^*, and $N(T)/N(0)$ may be obtained and compared with N_T^*/N_0^* (which is zero if N_T^* is zero).

The question might be raised: are there other forms of Lanchester's equations that might apply to the battle of Iwo Jima as well as equations

(1). The answer to this question is 'yes.' However, other equations that are fitted to these data require more complicated assumptions than the assumption that the A and B of equations (1) may be approximated by constants.

SUMMARY AND CONCLUSIONS

A technique has been shown for verifying the applicability of a certain type of generalized Lanchester equations to a certain kind of combat situation. This technique was applied in an analysis of the battle of Iwo Jima, in which it was found that the equations did describe the situation. The parameters characterizing this situation were determined.

When this technique has been applied to a large number of specific combat situations, it may be possible to draw conclusions of general interest and value. Such conclusions would be of the sort that assert the applicability (or nonapplicability) of specified types of generalized Lanchester equations to various classes of combat situation, determinations of the parameter values that pertain to these situations, and discoveries of 'operational constants,' or relationships, between these parameters and various other known factors pertaining to specific combat situations. Such relationships will be of particular value if it is possible to measure these other factors prior to the inception of a particular engagement. For example, it may be possible after sufficient research to ascertain the influence of such factors as terrain, time to prepare a defensive position, length and nature of artillery preparation and air support, ratio of combatant troops to troops in support, amount of troop experience, morale, etc., on the values of such parameters as A and B, the casualty producing rates of the forces about to enter into a specific battle.

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