Introduction to Series Practice Problems.

Solutions.

1. $a_n = \dfrac{2n}{3n+1}$

(a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$.

The sequence converges to $2/3$.

(b) Since $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

2. $\lim\limits_{n\to\infty} \dfrac{n^2}{5n^2+4} = \dfrac{1}{5} \neq 0$, so the series diverges by the Test for Divergence.

3. $\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

4. $\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = \lim_{k\to\infty} \frac{k^2+2k}{k^2+6k+9} = 1 \neq 0$, so the series diverges by the Test for Divergence.

5. $$\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$$

$2 = A(n+1) + B(n-1)$

$n=1$: $2=2A$ $A=1$ $n=-1$: $2=-2B$ $B=-1$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

S_n = sum of first n terms

= n^{th} partial sum.

$$S_2 = \left(\frac{1}{1} - \frac{1}{3}\right)$$

$$S_3 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$S_4 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$S_5 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right).$$

We observe the following pattern:

$$S_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = 1 + \frac{1}{2} - 0 - 0 = \frac{3}{2}.$$ The series converges to $\frac{3}{2}$.

6. $\frac{2}{n^2+4n+3} = \frac{2}{(n+3)(n+1)} = \frac{A}{n+3} + \frac{B}{n+1}$

$2 = A(n+1) + B(n+3)$

$n=-3: 2=-2A \quad A=-1 \quad n=-1: 2=2B \quad B=1$

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

S_n = n^{th} partial sum.

$S_1 = \left(\frac{1}{2} - \frac{1}{4}\right)$

$S_2 = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$

$S_3 = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$

$S_4 = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)$

We observe the following pattern:

$$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} - 0 - 0 = \frac{5}{6}.$$ The series converges to 5/6.

7. $$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$3 = A(n+3) + Bn$$

$n=0 \cdot A = 1$; $n=-3$ $B=-1$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

S_n = n^{th} partial sum.

$$S_1 = \left(\frac{1}{1} - \frac{1}{4}\right)$$

$$S_2 = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right)$$

$$S_3 = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right)$$

$$S_4 = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right).$$

We observe the following pattern:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\lim_{n \to \infty} S_n = 1 + \frac{1}{2} + \frac{1}{3} - 0 - 0 - 0 = \frac{11}{6}.$$

The series converges to $11/6$.

8. $\sum_{n=1}^{\infty} a_n$

$$S_n = \frac{n-1}{n+1}$$

$$S_1 = a_1 = \frac{1-1}{1+1} = 0 \Rightarrow a_1 = 0$$

$$S_2 = a_1 + a_2 = \frac{2-1}{2+1} = \frac{1}{3}$$

$$a_1 + a_2 = \frac{1}{3} \Rightarrow a_2 = \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3$$

$$= \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$$

$$a_1 + a_2 + a_3 = \frac{1}{2} \Rightarrow a_3 = \frac{1}{2} - 0 - \frac{1}{3} = \frac{1}{6}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{4-1}{4+1} = \frac{3}{5}$$

$$a_4 = \frac{3}{5} - \frac{1}{6} - \frac{1}{3} - 0 = \frac{1}{10}$$

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{n+1} = 1.}$$

The series converges to 1.

9. since $\sum_{n=1}^{\infty} a_n$ converges, we know that

$\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty} \frac{1}{a_n} = \infty \neq 0$

so $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.