Math 333
January 31, 2008
Linear Equations: A Second Approach
Integrating Factors for Linear Equations

Consider again the first-order linear differential equation

$$\frac{dy}{dt} = a(t)y + b(t). \tag{1}$$

Previously, we developed a guessing technique for solving such equations. Here, we'll develop a more general analytic technique.

1. First, rewrite the original DE in Eqn. (1) as

   $$\frac{dy}{dt} + g(t)y = b(t), \tag{2}$$

   where $g(t) = -a(t)$.

2. Note that the left-hand side of Eqn. (2) above *almost* looks like the product rule. Remember that the product rule says that

   $$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t).$$

3. Next, multiply both sides of the original DE in Eqn. (2) by $\mu(t)$. Then we have:

   $$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t). \tag{3}$$

   Note that the left-hand side of this equation looks even more like the derivative of a product.

4. Now, *assume* that we have found a function $\mu(t)$ such that

   $$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y.$$

   Then the DE in Eqn. (3) becomes

   $$\frac{d(\mu(t)y(t))}{dt} = \mu(t)b(t),$$

   which we can solve by integrating both sides:

   $$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt.$$

5. Conclusion: If we can fund a function $\mu(t)$ such that
$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y, \tag{4}$$
then we can solve the original differential equation.

6. So, now we must determine how to find such a function $\mu(t)$. If we rewrite Eqn. (4) above using the product rule, we obtain:
$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y. \tag{5}$$

7. Thus, $\mu(t)$ must satisfy
$$\frac{d\mu}{dt} = \mu(t)g(t),$$
which we can rewrite as
$$\mu(t) = e^{\int g(t)\,dt}.$$

Summary. To solve the differential equation
$$\frac{dy}{dt} + g(t)y = b(t),$$
perform the following steps.

1. Compute the integrating factor
$$\mu(t) = e^{\int g(t)\,dt}.$$

2. Multiply both sides of the original DE by $\mu(t)$ and integrate. Then
$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt.$$

Note. This method requires the calculation of two integrals, so we may not, in general, be able to find a general formula for $y(t)$ that does not contain integrals.

Example. Find the general solution of
$$\frac{dy}{dt} = -\frac{y}{t} + 2.$$

Example. Find the general solution of
$$\frac{dy}{dt} = (\sin t)y + 4.$$