Improper Integrals

There are two types of improper integrals that we will consider: integrals that involve **infinite intervals** and integrals that involve a **discontinuous (unbounded) integrand**.

Infinite Intervals

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

In part (c) any real number a can be used.

Examples.

1. Determine whether the integral $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent or divergent. If it is convergent, evaluate the integral. Interpret this result in terms of area.
2. Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent. If it is convergent, evaluate the integral. Interpret this result in terms of area.
3. Determine whether the integral $\int_1^\infty \frac{4}{x^3}\,dx$ is convergent or divergent. If it is convergent, evaluate the integral. Interpret this result in terms of area.

4. **The p-test for convergence of integrals.** For which (if any) values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?
5. Evaluate $\int_{-\infty}^{1} 2e^{2x}\,dx$.
6. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Integrands that contain a discontinuity (unbounded integrand)

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx,$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx,$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Examples.

Determine whether each integral is convergent or divergent. If it is convergent, evaluate the integral.

1. $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$
2. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$
3. $\int_0^3 \frac{1}{x-1}\,dx$
4. Find the area of the region $S = \{(x, y) : \; -2 < x \leq 0 \text{ and } 0 \leq \frac{1}{\sqrt{x+2}}\}$.