Why does the harmonic series diverge?

We have seen in class (using the Integral Test or the p-Test for Series) that the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. This is a particularly interesting result since $\lim_{k\to\infty} \frac{1}{k} = 0$, i.e. we have a sequence $\{a_k\}$ such that $\lim_{k\to\infty} a_k = 0$ but $\sum_{k=1}^{\infty} a_k$ diverges. This means that

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

becomes infinitely large. The following argument illustrates why this happens. We'll look at the partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and show that $S_n \to \infty$ as $n \to \infty$.

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= 1 + \frac{3}{3}
\end{aligned}$$

Similarly, $S_{16} > 1 + \frac{4}{2}$, $S_{32} > 1 + \frac{5}{2}$, $S_{64} > 1 + \frac{6}{2}$, and in general,

$$S_{2^n} > 1 + \frac{n}{2}.$$

Thus, $S_{2^n} \to \infty$ as $n \to \infty$, so the series diverges.