Geometric Series

Definition. A **geometric series** is a series of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots ,$$

where a and r are real numbers. The number a is called the **leading term** and r is called the **ratio**.

Example 1. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$ is a geometric series with leading term 1 and ratio 1/2.

Partial sums of geometric series. Consider the geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots .$$

The partial sum S_n is given by

$$S_n = a + ar + ar^2 + \cdots + ar^n.$$

Then

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n+1}.$$

Subtracting these equations, we obtain

$$S_n - rS_n = a - ar^{n+1}.$$

Thus,

$$S_n(1 - r) = a(1 - r^{n+1}),$$

which we can simply to obtain a formula for S_n:

$$S_n = a\frac{1 - r^{n+1}}{1 - r}.$$

We can use this formula to determine whether or not a given geometric series converges. Recall that a series converges if $\lim_{n\to\infty} S_n$ exists. For the geometric series, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1 - r^{n+1}}{1 - r}.$$

Recall that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } |r| > 1 \text{ or } r = -1 \end{cases}$$

Thus, we can conclude the following:

$$\lim_{n\to\infty} S_n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{does not exist} & \text{if } |r| \geq 1 \end{cases}$$

Theorem: Convergence and divergence of geometric series. The geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

:

- **converges** to $\dfrac{a}{1-r}$ if $|r| < 1$ and
- **diverges** if $|r| \geq 1$

Example 2. Use the geometric series formula to show that

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

converges and is equal to 2.

Example 3. Does the series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

converge or diverge? If the series converges, find its value.

Example 4. Does the series

$$\sum_{k=0}^{\infty} \frac{1+2^n}{3^n}$$

converge or diverge? If the series converges, find its value.

Example 5. Does the series

$$\sum_{n=0}^{\infty} 2^{2n} 3^{1-n}$$

converge or diverge? If the series converges, find its value.