Putnam Problem-Solving Seminar
Tuesday, November 6, 2007

There are too many problems to consider each one in one session alone. Just pick a few problems that you like and try to solve them. However, you are not allowed to work on a problem that you already know how to solve.

Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra.

Geometric Series

- $\displaystyle\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$, for $x \neq 1$
- $\displaystyle\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, for $|x| < 1$

Problems.

1. Evaluate
$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}.$$
2. What is the value of c if $\displaystyle\sum_{n=2}^{\infty}(1+c)^{-n} = 2$?
3. Sum the finite series $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$.
4. Evaluate $\displaystyle\sum_{(p,q)=1} \frac{1}{x^{p+q}-1}$, $|x| > 1$, where the sum is taken overall positive integers p and q such that $gcd(p,q) = 1$.
5. Sum the series $1 + 22 + 333 + \ldots + n(11\ldots1)$, where $(11\ldots1)$ contains n ones.
6. Verify the following formulas:
   (a) $\displaystyle\sum_{i=1}^{n} \sin((2k-1)\theta) = \frac{\sin^2(n\theta)}{\sin\theta}$
   (b) $\displaystyle\sum_{i=1}^{n} \sin^2((2k-1)\theta) = -\frac{1}{2}n - \frac{\sin(4n\theta)}{4\sin 2\theta}$

7. (a) If one tosses a fair coin until a first head appears, what is the probability that this event occurs on an even-numbered toss?

   (b) The game of craps is played in the following manner: A player tosses a pair of dice. If the number is 2, 3, or 12, he loses immediately. If it is 7 or 11, he wins immediately. If any other number is obtained on the first toss, then that number becomes the players point, and he must keep tossing the dice until either he makes the point (i.e. obtains the first number again), in which case he wins, or he obtains 7, in which case he loses. Find the probability of winning.

8. If a, b, and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$, show that a, b, and c are distinct. Then show that
$$\frac{a^{1000} - b^{1000}}{a - b} + \frac{b^{1000} - c^{1000}}{b - c} + \frac{c^{1000} - a^{1000}}{c - a}$$
is an integer.

9. Prove that $\Pi_{n-0}^{\infty}(1 + x^{2n}) = \displaystyle\sum_{n=0}^{\infty} x^n$, $|x| < 1$.

Telescoping Series Problems.

1. Sum the infinite series
$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}.$$

2. Sum the infinite series
$$\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}.$$

3. Sum the infinite series
$$\sum_{n=1}^{\infty} (\frac{3}{n(n+1)} + \frac{1}{2^n}).$$

4. Sum the infinite series
$$\frac{3}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \ldots$$

5. Express
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + m n^2 + 2mn}$$
as a rational number.

6. Sum the series
$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3(\frac{x}{3^n}).$$

7. Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ be the Fibonacci sequence. Prove the following identities:

   (a) $F_1 + F_2 + \ldots F_n = F_{n+2} - 1$.

   (b) $F_1 + F_3 + F_{2n-1} = F_{2n}$.

   (c) $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$.

8. Let
$$d_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$
Show that
$$d_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}.$$