Math 333
Fundamental Solutions
Second-Order Linear Differential Equations
Thursday, February 14, 2008

We've talked about how to solve *some* differential equations of the form

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are constants. In particular, we've learned how to find the general solution of the DE in Eqn. (1) when the characteristic equation

$$ar^2 + br + c = 0$$

has **real and distinct** roots (also called eigenvalues, or characteristic values).

Next, we'll consider the mathematical details regarding second order linear homogeneous differential equations in more generality. We'll use the theory that we develop here to construct solutions of the second-order homogeneous linear differential equation with constant coefficients for the cases in which the characteristic equation has either a pair of complex conjugate roots or a real double root.

We'll begin with an existence and uniqueness theorem. Given an arbitrary second-order linear differential equation, it's natural to ask whether or not we are certain that a solution must exist. Moreover, if a solution does exist, are we guaranteed that it's the only one? The existence and uniqueness theorem addresses these questions.

Existence and Uniqueness Theorem (EUT) for Second-Order Linear Differential Equations. Consider the initial-value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

where $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval I that contains the point t_0. Then there is **exactly one solution** $y(t)$ of this problem, and the solution exists throughout the interval I.

Example 1. Find the longest interval in which the solution of the initial-value problem

$$(t^2 - 3t)y'' + ty' - (t + 3)y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist. **Solution.** The longest open interval containing the initial point $t = 1$ in which the solution is guaranteed to exist is $0 < t < 3$.

Example 2. Find the unique solution of the IVP

$$y'' + t^2y' + \sin(t)y = 0, \quad y(3) = 0, \quad y'(3) = 0.$$

Solution. The function $y(t) = 0$ satisfies the DE and the initial conditions. By the Uniqueness Theorem, it is the only solution of the given IVP.

The Linearity Principle. Suppose that y_1 and y_2 are two solutions of the **homogeneous** differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Then the **linear combination**

$$y(t) = k_1 y_1 + k_2 y_2$$

is also a solution for any values of the constants k_1 and k_2.

Proof. Check that $y(t) = k_1 y_1 + k_2 y_2$ indeed satisfies the DE.

Definition. Consider the general second order linear homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{2}$$

Suppose that y_1 and y_2 are solutions of Eqn. (2). The **Wronskian** of the solutions y_1 and y_2 is the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

We need the Wronskian for the following result.

Theorem. Suppose that y_1 and y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = 0$$

and that there is a point t_0 where the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

is not zero. Then the family of solutions

$$y(t) = k_1 y_1(t) + k_2 y_2(t)$$

with arbitrary constants k_1 and k_2 includes every solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

This theorem states that as long as the Wronskian of y_1 and y_2 is not everywhere zero, then the linear combination

$$y(t) = k_1 y_1 + k_2 y_2$$

is the **general solution** of the second-order linear homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Definition. The solutions y_1 and y_2 with a nonzero Wronskian are said to form a **fundamental set of solutions** for the DE $y'' + p(t)y' + q(t)y = 0$.

Example 1. Suppose that r_1 and r_2 are two real, distinct roots of the characteristic equation $ar^2 + br + c = 0$. Show that $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ form a fundamental set of solutions for the second order homogeneous linear DE with constant coefficients $ay'' + by' + cy = 0$.

Example 2. Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \quad t > 0.$$

Summary. We can summarize the discussion in this section as follows. Suppose that we want to find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad \alpha < t < \beta.$$

We first find two functions y_1 and y_2 that satisfy the DE for $\alpha < t < \beta$. Then, we must make sure that there is a point in the interval $[\alpha, \beta]$ where the Wronskian $W(y_1, y_2)$ is nonzero (i.e. we must check that the Wronskian $W(y_1, y_2)$ is not identically zero). Then y_1 and y_2 form a fundamental set of solutions of the DE and the general solution is

$$y(t) = k_1 y_1(t) + k_2 y_2(t),$$

where k_1 and k_2 are arbitrary constants.