Math 333
Qualitative Results: Forced Harmonic Oscillators

Forced Harmonic Oscillators. Recall our derivation of the second-order linear homogeneous differential equation with constant coefficients:

$$my'' = -ky - by'. \tag{1}$$

We obtained this differential equation by considering the motion of a mass $m > 0$ attached to a spring sliding on a table, where $k > 0$ represents the spring constant and $b > 0$ represents the damping force. For ease of notation, we often rewrite this differential equation as

$$y'' + py' + qy = 0, \tag{2}$$

where $p = b/m > 0$ and $q = k/m > 0$. Using the method of eigenvalues (roots of the characteristic equation), we have learned how to solve any differential equation of the form given by Eqn. (2).

To include an external force in the mass-spring system, we must add another term to the right-hand side of Eqn. (1). We consider external forcing functions that depend only on time t, so the external force is given by a function $f(t)$. The new equation is then

$$my'' = -ky - by' + f(t),$$

which we can rewrite as

$$y'' + \frac{b}{m}y' + \frac{k}{m}y = \frac{f(t)}{m}.$$

Finally, for convenience of notation, we rewrite the previous equation as

$$y'' + py' + qy = g(t), \tag{3}$$

where $p = b/m > 0$, $q = k/m > 0$, and $g(t) = f(t)/m$. The differential equation in Eqn. (3) is a second-order linear constant-coefficient non-homogeneous differential equation. We will also refer to this equation as a **forced equation** or **forced harmonic oscillator**, and we will refer to $g(t)$ as the **forcing function**. The differential equation

$$y'' + py' + qy = 0 \tag{4}$$

is the **associated homogeneous** or **unforced** equation.

Previously, we have determined how to solve the forced equation by solving the associated homogeneous equation and using the Extended Linearity Principle. More precisely, if

$$k_1 y_1(t) + k_2 y_2(t)$$

is the general solution of the associated homogeneous equation, and if

$$y_p(t)$$

is one particular solution of the forced equation, then the general solution of the forced equation is

$$y(t) = k_1 y_1(t) + k_2 y_2(t) + y_p(t).$$

Next, we will study **qualitative results** about solutions of forced harmonic oscillators.

1. (Real and distinct roots of the characteristic equation.) Consider the forced equation $y'' + 3y' + 2y = e^{-3t} + 4$.

   (a) Simulate the differential equation graphically for the initial conditions described below by converting the second-order differential equation to a system of first-order differential equations. It may be useful to use the Maple file ForcedOscillator.mw posted on the P: drive (and linked on the Course Schedule page) as a reference. For each set of initial conditions, describe the long term behavior of the graph of $y(t)$ as $t \to \infty$.

      i. $y(0) = 0, \quad y'(0) = 0$
      ii. $y(0) = 2, \quad y'(0) = -3$
      iii. $y(0) = -4, \quad y'(0) = 1$
      iv. $y(0) = -10, \quad y'(0) = 10$

   (b) Does the long term behavior of the position of the mass depend on the initial conditions?

   (c) Find the general solution $y_h(t)$ of the associated homogeneous equation, and compute $\lim_{t \to \infty} y_h(t)$. Does this computation depend on the values of the constants k_1 and k_2?

   (d) Find a particular solution $y_p(t)$ of the forced equation, and compute $\lim_{t \to \infty} y_p(t)$.

   (e) Find the general solution $y(t)$ of the forced equation, and compute $\lim_{t \to \infty} y(t)$. Does this computation depend on the values of the constants k_1 and k_2? Discuss.

2. (Complex roots of the characteristic equation.) Consider the forced equation $y'' + 2y' + 2y = \sin t$.

   (a) Simulate the differential equation graphically for the initial conditions described below by converting the second-order differential equation to a system of first-order differential equations. It may be useful to use the Maple file ForcedOscillator.mw posted on the P: drive (and linked on the Course Schedule page) as a reference. For each set of initial conditions, describe the long term behavior of the graph of $y(t)$ as $t \to \infty$.

      i. $y(0) = 0, \quad y'(0) = 0$

     ii. $y(0) = 2, \quad y'(0) = -3$

    iii. $y(0) = -4, \quad y'(0) = 1$

    iv. $y(0) = -10, \quad y'(0) = 10$

(b) Does the long term behavior of the position of the mass depend on the initial conditions?

(c) Find the general solution $y_h(t)$ of the associated homogeneous equation, and compute $\lim\limits_{t \to \infty} y_h(t)$. Does this computation depend on the values of the constants k_1 and k_2?

(d) Find a particular solution $y_p(t)$ of the forced equation.

(e) Find the general solution $y(t)$ of the forced equation, and discuss the behavior of $y(t)$ as $t \to \infty$. Does this behavior depend on the values of the constants k_1 and k_2?

3. (Real repeated roots of the characteristic equation.) Consider the forced equation $y'' + 4y' + 4y = 2t$.

(a) Simulate the differential equation graphically for the initial conditions described below by converting the second-order differential equation to a system of first-order differential equations. It may be useful to use the Maple file ForcedOscillator.mw posted on the P: drive (and linked on the Course Schedule page) as a reference. For each set of initial conditions, describe the long term behavior of the graph of $y(t)$ as $t \to \infty$.

      i. $y(0) = 0, \quad y'(0) = 0$

     ii. $y(0) = 2, \quad y'(0) = -3$

    iii. $y(0) = -4, \quad y'(0) = 1$

    iv. $y(0) = -10, \quad y'(0) = 10$

(b) Does the long term behavior of the position of the mass depend on the initial conditions?

(c) Find the general solution $y_h(t)$ of the associated homogeneous equation, and compute $\lim\limits_{t \to \infty} y_h(t)$. Does this computation depend on the values of the constants k_1 and k_2?

(d) Find a particular solution $y_p(t)$ of the forced equation.

(e) Find the general solution $y(t)$ of the forced equation, and discuss the behavior of $y(t)$ as $t \to \infty$. Does this behavior depend on the values of the constants k_1 and k_2?

General results. Consider the general unforced harmonic oscillator

$$y'' + py' + qy = 0, \tag{5}$$

where $p > 0$ and $q > 0$.

1. Simulate the second-order differential equation for various positive values of the constants p and q, and for various initial conditions. Discuss the long term behavior of $y(t)$. Does the long term behavior of $y(t)$ depend on the values of the constants p and q or on the initial conditions?

2. Find the general solution $y(t)$ of the unforced equation. Note that you will have to consider three separate cases, depending on whether the roots of the characteristic equation
$$r^2 + pr + q = 0$$
are real and distinct, complex, or real repeated.

   (a) Suppose that $p^2 - 4q > 0$. Find the general solution $y(t)$ of Eqn. (5). Discuss the behavior of $y(t)$ as $t \to \infty$.

   (b) Suppose that $p^2 - 4q < 0$. Find the general solution $y(t)$ of Eqn. (5). Discuss the behavior of $y(t)$ as $t \to \infty$.

   (c) Suppose that $p^2 - 4q = 0$. Find the general solution $y(t)$ of Eqn. (5). Discuss the behavior of $y(t)$ as $t \to \infty$.

3. Conclude. For any $p > 0$, $q > 0$, and for any initial conditions, describe the behavior of the general solution $y(t)$ of

$$y'' + py' + qy = 0$$

   as $t \to \infty$.

4. Finally, consider the general forced harmonic oscillator

$$y'' + py' + qy = g(t), \tag{6}$$

   where $p > 0$ and $q > 0$. Let $y_h(t)$ denote the general solution of the associated homogeneous (unforced) equation, and let $y_p(t)$ denote a particular solution of the forced equation. Let $y(t)$ denote the general solution of the forced equation:

$$y(t) = y_h(t) + y_p(t).$$

   Discuss the behavior of the general solution $y(t)$ as $t \to \infty$.

Conclusion: Consider the forced harmonic oscillator equation

$$y'' + py' + qy = g(t),$$

where p and q are positive constants. Let $y_h(t)$ denote the general solution of the associated homogeneous (unforced) equation

$$y'' + py' + qy = 0$$

and let $y_p(t)$ denote a particular solution of the forced equation. In applications, $y_h(t)$ is called the **natural response** and $y_p(t)$ is called the **forced response**. Let $y(t)$ denote the general solution of the forced equation:

$$y(t) = y_h(t) + y_p(t).$$

As $t \to \infty$, $y_h(t)$ tends to zero. Thus, *for large t,*

$$y(t) \approx y_p(t).$$

The behavior of the forced response $y_p(t)$ for large t is called the **steady-state response**. The discussion above can be restated as follows: *All solutions of a forced, damped harmonic oscillator approach the steady-state response because the natural response dies out, leaving only the forced response.*