Math 333
January 24, 2008
Existence and Uniqueness Theorems

The General Setup. Consider the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

The Existence Theorem. Suppose that $f(t, y)$ is *continuous* in a rectangle of the t, y-plane containing the initial point (t_0, y_0). Then, there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t_0 < t_0 + \epsilon$ that solves the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

In practice, this means that as long as our differential equation $\frac{dy}{dt} = f(t, y)$ is *nice* in a region of the initial point (t_0, y_0), then we are guaranteed to have a solution of the initial-value problem. However, we must be careful about the domain of definition of our solution function $y(t)$. The solution function $y(t)$ might not be defined for all values of t.

The Uniqueness Theorem. Suppose that $f(t, y)$ and $\frac{\partial f}{\partial y}$ (the partial derivative of f with respect to y) are *both continuous* in a rectangle of the t, y-plane containing the initial point (t_0, y_0). Then, the solution of the IVP is unique. This means that if $y_1(t)$ and $y_2(t)$ are two functions that solve the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (where ϵ is some positive number), then

$$y_1(t) = y_2(t).$$

In practice, this means that if we have the additional structure that *both* $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous, then we know that there is exactly one solution of the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

In class today, we'll look at some examples of situations in which we can use the Uniqueness Theorem to obtain qualitative information about solutions of IVP's.