Euler Equations

A relatively simple differential equation that has a regular singular point is the **Euler equation**,

$$x^2y'' + \alpha xy' + \beta y = 0, \tag{1}$$

where α and β are real constants. We can check that $x = 0$ is a regular singular point of Eqn. (1). Because the solution of the Euler equation is typical of the solutions of all differential equations with a regular singular point, we'll consider the Euler equation in detail before we consider the general problem of finding series solutions of a differential equation near a regular singular point.

By the Existence Theorem, we know that in any interval not containing $x = 0$, Eqn. (1) has a general solution of the form $y(x) = c_1y_1(x) + c_2y_2(x)$, where y_1 and y_2 are linearly independent solutions and c_1 and c_2 are arbitrary constants. First, we'll consider the interval $x > 0$. Later, we'll extend our results to the interval $x < 0$.

Since

$$(x^r)' = rx^{r-1} \text{ and } (x^r)'' = r(r-1)x^{r-2},$$

it is natural to guess that Eqn. (1) has a solution of the form

$$y = x^r.$$

Substituting $y = x^r$ into the differential equation, we obtain

$$x^r[r(r-1) + \alpha r + \beta] = 0.$$

If r is a root of the quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0,$$

then $x^r[r(r-1) + \alpha r + \beta] = 0$, and $y = x^r$ is a solution of Eqn. (1). The roots of the quadratic equation $F(r)$ are given by

$$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2},$$

and $F(r) = (r - r_1)(r - r_2)$. As for second order linear equations with constant coefficients, we must consider separately the cases in which the roots are real and unequal, real and equal, and complex conjugates. Our discussion here will be similar to our discussion of second order linear differential equations with constant coefficients, with e^{rx} replaced by x^r (and you will have an exam question addressing this similarity).

Case 1: Real, distinct roots. Suppose that $F(r) = 0$ has real roots $r_1 \neq r_2$. Then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions of Eqn. (1). Since

$$W(y_1, y_2) = (r_2 - r_1)x^{r_1+r_2-1} \neq 0,$$

it follows that the general solution of Eqn. (1) is

$$y = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0. \tag{2}$$

Recall that if r is not rational, then x^r is defined by $x^r = e^{r \ln x}$.

Example 1. Solve the differential equation

$$2x^2 y'' + 3xy' - y = 0, \quad x > 0.$$

Solution. The general solution is

$$y = c_1 x^{1/2} + c_2 x^{-1}, \quad x > 0.$$

Case 2: Real, equal roots. Suppose that $F(r) = 0$ has two real roots $r_1 = r_2$. Then we obtain only one solution of the form $y_1 = x^{r_1}$. We can use the method of *reduction of order* to find a second solution of Eqn. (1). We guess a solution of the form $y = vy_1 = vx^{r_1}$, where v is some unknown function of x. Substituting into Eqn. (1), and using the fact that $2r_1 + \alpha = 1$ since $r_1 = -(\alpha - 1)/2$ is a double root of $F(r) = 0$, we find that $v = c_1 + c_2 \ln x$, so a second solution of Eqn. (1) is $y_2(x) = x^{r_1} \ln x$. Since

$$W(y_1, y_2) = x^{2r_1 - 1} \neq 0,$$

x^{r_1} and $x^{r_1} \ln x$ form a fundamental set of solutions. Thus, the general solution of Eqn. (1) in this case is

$$y(x) = (c_1 + c_2 \ln x)x^{r_1}. \tag{3}$$

Example 2. Solve the differential equation

$$x^2 y'' + 5xy' + 4y = 0, \quad x > 0.$$

Solution. The general solution is

$$y = (c_1 + c_2 \ln x)x^{-2}, \quad x > 0.$$

Case 3: Complex Roots. Finally, suppose that the roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ are complex conjugates with $\mu \neq 0$. We must now consider what is meant by x^r when r is complex. Recall that x^r is defined by $x^r = e^{r \ln x}$. Thus,

$$\begin{aligned} x^{\lambda+i\mu} &= e^{(\lambda+i\mu)\ln x} \\ &= e^{\lambda \ln x} e^{i\mu \ln x} \\ &= x^{\lambda}[\cos(\mu \ln x) + i \sin(\mu \ln x)]. \\ x^{\lambda-i\mu} &= x^{\lambda}[\cos(\mu \ln x) - i \sin(\mu \ln x)]. \end{aligned}$$

As we did in the case of second order linear differential equations with constant coefficients, we observe that $y_1 = x^{\lambda}\cos(\mu \ln x)$ and $y_2 = x^{\lambda}\sin(\mu \ln x)$ are also solutions of Eqn. (1). Since

$$W(y_1, y_2) = \mu x^{2\lambda-1} \neq 0,$$

y_1 and y_2 form a fundamental set of solutions, so the general solution of Eqn. (1) in this case is

$$y(x) = c_1 x^{\lambda}\cos(\mu \ln x) + c_2 x^{\lambda}\sin(\mu \ln x). \tag{4}$$

Example 3. Solve the differential equation

$$x^2 y'' + xy' + y = 0, \quad x > 0.$$

Solution. The general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x), \quad x > 0.$$

Finally, we consider how to extend the solutions to the interval $x < 0$. The solutions of the Euler equation that we have constructed for $x > 0$ can be shown to be valid for $x < 0$, but in general they are complex-valued. For example, in the solution to Example 1, $x^{1/2}$ is imaginary for $x < 0$. We can construct real-valued solutions in the interval $x < 0$ in a straightforward way by making the following change of variable. Let $x = -\gamma$, where $\gamma > 0$, and let $y = u(\gamma)$. Then:

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{d\gamma}\frac{d\gamma}{dx} \\ &= -\frac{du}{d\gamma} \\ \frac{d^2y}{dx^2} &= \frac{d}{d\gamma}\left(-\frac{du}{d\gamma}\right)\frac{d\gamma}{dx} \\ &= \frac{d^2u}{d\gamma^2}. \end{aligned}$$

Thus, for $x < 0$, Eqn. (1) has the form

$$\gamma^2 u'' + \alpha\gamma u' + \beta u = 0, \quad \gamma > 0.$$

But this is precisely the problem that we have already solved. From our previous work, we obtain:

$$u(\gamma) = \begin{cases} c_1\gamma^{r_1} + c_2\gamma^{r_2} & \text{if } r_1, r_2 \in \mathbb{R},\ r_1 \neq r_2 \\ (c_1 + c_2 \ln\gamma)\gamma^{r_1} & \text{if } r_1, r_2 \in \mathbb{R},\ r_1 = r_2 \\ c_1\gamma^{\lambda}\cos(\mu\ln\gamma) + c_2\gamma^{\lambda}\sin(\mu\ln\gamma) & \text{if } r_1, r_2 = \lambda \pm i\mu. \end{cases}$$

To obtain u in terms of x, we replace γ by $-x$ in the previous expressions. Finally, we can combine the results for $x > 0$ and $x < 0$ by recalling that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$. Thus, we need only replace x by $-x$ in Eqns. (2), (3), and (4) to obtain real-valued solutions valid in any interval not containing $x = 0$.

Summary. The general solution of the Euler equation (1)

$$x^2y'' + \alpha xy' + \beta y = 0$$

in any interval not containing $x = 0$ is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

- If the roots r_1 and r_2 are real and distinct, then

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}.$$

- If the roots r_1 and r_2 are real and equal, then

$$y(x) = (c_1 + c_2\ln|x|)|x|^{r_1}.$$

- If the roots $r_1, r_2 = \lambda \pm i\mu$ are complex conjugates, then

$$y(x) = |x|^{\lambda}[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)].$$