Math 112
Approximating Integrals Numerically: Error Bounds

Consider the definite integral

$$I = \int_a^b f(x)\,dx.$$

For any integer n, use the following notation:

- L_n denotes the left rectangular sum approximation using n rectangles.
- R_n denotes the right rectangular sum approximation using n rectangles.
- M_n denotes the midpoint rectangular sum approximation using n rectangles.
- T_n denotes the trapezoid approximation using n trapezoids.

Previously, we've thought about how the various approximating methods are related to one another (for certain types of functions), and how those relationships can be used to obtain bounds on the numerical value of I. Now, we'll think about *error bounds for approximating sums.* The error bound theorems provide a bound on the *maximum error* that we can have when we use approximating methods to estimate an integral. More importantly, we'll use the error bound theorems to determine how many sub-intervals a given method needs to approximate an integral with specified accuracy.

1. **Error bounds for L_n and R_n.** To find the maximum error committed by the left and right rectangular sum approximations, first find a constant K_1 such that

   $$|f'(x)| \le K_1 \text{ for all } x \text{ in } [a, b].$$

   Then

   $$|I - L_n| \le \frac{K_1(b-a)^2}{2n}$$

   and

   $$|I - R_n| \le \frac{K_1(b-a)^2}{2n}.$$

2. **Error bounds for M_n and T_n.** To find the maximum error committed by the midpoint and trapezoid sum approximations, first find a constant K_2 such that

   $$|f''(x)| \le K_2 \text{ for all } x \text{ in } [a, b].$$

Then

$$|I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$$

and

$$|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2}.$$

Some useful properties of absolute value.

- $|AB| = |A||B|$
- (Triangle Inequality) $|A + B| \leq |A| + |B|$

Examples.

1. Consider the integral $I = \displaystyle\int_2^5 \sin(x^2)\, dx$.
   (a) Find the maximum error committed by L_8 in approximating I.
   (b) Find the maximum error committed by M_8 in approximating I.
2. Consider the integral $I = \displaystyle\int_1^5 \cos(\frac{1}{x})\, dx$.
   (a) How large does n need to be to use right rectangular sums to approximate I with error guaranteed to be less than 0.00001?
   (b) How large does n need to be to use trapezoid sums to approximate I with error guaranteed to be less than 0.00001?