Math 347
Discrete Dynamical Systems of Several Equations

Example. Consider the following study of population growth. Consider a species which can be broken into three equal age groups: 0-1 years, 1-2 years, and 2-3 years. Note that instead of years, we can use decades or other units as needed for a particular situation. Let $a(n)$ represent the number of individuals in the 0-1 age group in year n; let $b(n)$ represent the number of individuals in the 1-2 age group in year n; and let $c(n)$ represent the number of individuals in the 2-3 age group in year n. Suppose that we know the following information about the species:

- Half of the individuals in the 0-1 age group survive one year.
- Two-thirds of the individuals in the 1-2 age group survive to the next year.
- On average, each individual in the 0-1 age group has 0.5 offspring.
- On average, each individual in the 1-2 age group has 5 offspring.
- On average, each individual in the 2-3 age group has 3 offspring.

We can construct a dynamical system of three equations to model this population.

Some important properties of dynamical systems of several equations:

- If $A(n+1) = RA(n)$ is a dynamical system of m equations, where R is an $m \times m$ matrix, then the solution of the dynamical system is

$$A(n) = R^n A(0).$$

- This is not a useful solution form since matrix multiplication is computationally difficult and time-consuming.
- Instead, we use eigenvalues and eigenvectors. Suppose that we have an $m \times m$ matrix R, an m-vector B, and that we wish to compute $R^k B$. Then we follow these four steps:

  1. Find the eigenvalues of R, i.e. solve the polynomial $p(\lambda) = \det(R - \lambda I) = 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ denote the eigenvalues of R.
  2. Find the corresponding eigenvectors of R. Let $\mathbf{v_1}$, $\mathbf{v_2}, \ldots, \mathbf{v_m}$ denote the eigenvectors of R.
  3. Write the vector B as a linear combination of the eigenvectors of R:

$$B = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_m\mathbf{v_m}.$$

4. Compute R^kB:

$$\begin{aligned} R^kB &= R^k(x_1\mathbf{v_1} + x_2\mathbf{v_2} + \ldots + x_m\mathbf{v_m}) \\ &= x_1\lambda_1^k\mathbf{v_1} + x_2\lambda_2^k\mathbf{v_2} + \ldots + x_m\lambda_m^k\mathbf{v_m} \end{aligned}$$

- Consider an $m \times m$ matrix R with m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. We define the absolute value of R to be

$$||R|| = \text{maximum}\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|\}.$$

If $||R|| < 1$, then

$$\lim_{k\to\infty} A(k) = \mathbf{0},$$

where $\mathbf{0}$ means the vector with m zeros.

Problems. These problems are due on Tuesday, November 6.

1. Find the particular solution to the system of equations

$$\begin{aligned} a(n+1) &= a(n) + 2b(n) \\ b(n+1) &= -a(n) + 4b(n) \end{aligned}$$

with initival values $a(0) = 1$ and $b(0) = 2$. Discuss the long-term behavior of the system.

2. Find the particular solution to the system of equations

$$\begin{aligned} a(n+1) &= -0.5a(n) + 0.5b(n) \\ b(n+1) &= -0.5a(n) + 0.75b(n) \end{aligned}$$

with initival values $a(0) = 6$ and $b(0) = 6$. Discuss the long-term behavior of the system.