Introduction to Dynamical Systems

A **dynamical system** consists of n state variables $(x_1, x_2, \ldots, x_n)$ and a system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ &\cdots \\ &\cdots \\ &\cdots \\ \frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n), \end{aligned}$$

defined on the *state space* S, where S is some subset of $\mathbf{R}^n$. The points at which $\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}$ are all equal to zero are called the **equilibrium points** of the dynamical system. The **nullclines** of a dynamical system are the set of solution curves of

$$\begin{aligned} \frac{dx_1}{dt} &= 0 \\ &\cdots \\ &\cdots \\ &\cdots \\ \frac{dx_n}{dt} &= 0. \end{aligned}$$

The equilibrium points occur at the intersection of the nullclines in the state space S. We will be interested in discussing the **stability** of equilibrium points.

Example. Consider the dynamical system given by

$$\begin{aligned} \frac{dx}{dt} &= 2x(1 - \tfrac{x}{2}) - xy \\ \frac{dy}{dt} &= 3y(1 - \tfrac{y}{3}) - 2xy \end{aligned},$$

where x and y are two species that compete for resources.

Example. Ecologists use the following model to represent the growth processes of two competing species x and y:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x(1 - \tfrac{x}{K_1}) - \alpha_1 xy \\ \frac{dy}{dt} &= r_2 y(1 - \tfrac{y}{K_2}) - \alpha_2 xy \end{aligned},$$

where the parameters r_i represent the intrinsic growth rates of each species; K_i represents the carrying capacity of each species; and α_i represents the effects of competition.

The blue whale and fin whale are two similar species that inhabit the same areas and compete for resources. The intrinsic growth rate of each species is estimated to be 5% for the blue whale and 8% for the fin whale (per year). The environmental carrying capacity for each species is estimated at 150,000 blue whales and 400,000 fin whales. Suppose that the effect of competition between the two species is given by the parameter $\alpha = \alpha_1 = \alpha_2 = 10^{-7}$. Further, suppose that intense harvesting has reduced the whale population to around 5,000 blue whales and 70,000 fin whales. Will the blue whale become extinct?