Math 347
(Review of) Discrete Probability Models

A discrete random variable X is a variable that can assume a countable number of values, say $x_1, x_2, \ldots$. Thus, without loss of generality, we can assume that the values that the random variable can assume are taken from the integers. Examples of a discrete random variable include the number of people waiting in line at a bank at a given time, the number of defective compact discs in a group of 100 drawn from a production line at a given time, or the number of robberies in a police district in a year.

The probability distribution associated with a **discrete random variable**

$$X \in \Omega = \{x_1, x_2, \ldots\}$$

is denoted p_i and is the probability that X equals x_i, i.e.

$$\mathbb{P}(X = x_i) = p_i.$$

The probability distribution must satisfy the following:

- $\sum_i p_i = 1$.
- $0 \leq p_i \leq 1$ for all i.

The Poisson, Bernoulli, binomial, and geometric probability distributions are among the most commonly used discrete probability distributions in mathematical modeling. The mean, or average, or **expected value** of X is given by

$$\mathbb{E}[X] = \sum_i x_i p_i.$$

The **variance** of X measures the extent to which X deviates from the mean, and is given by

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Note that for a discrete random variable X, this is equivalent to

$$\mathrm{Var}[X] = \sum_i (x_i - \mathbb{E}[X])^2 p_i.$$

The probability that $a \leq X \leq b$ is given by

$$\mathbb{P}(a \leq X \leq b) = \sum_{i=a}^{b} p_i.$$

Suppose that

$$Y \in \Omega_1 = \{y_1, y_2, y_3, \ldots\}$$

and

$$Z \in \Omega_2 = \{z_1, z_2, z_3, \ldots\}$$

are two discrete random variables. We say that Y and Z are **independent** if

$$\mathbb{P}(Y = y_i \text{ AND } Z = z_j) = \mathbb{P}(Y = y_i)\mathbb{P}(Z = z_j).$$

More generally, two *events* A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

For a discrete random variable $X \in \Omega = \{x_1, x_2, \ldots\}$, an event is just a subset of Ω. The probability of A or B is

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Note that if events A and B are mutually exclusive, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Conditional probability is the probability that some event, say A, occurs, given that some other event, say B, has already occurred. Conditional probability is written

$$\mathbb{P}(A|B),$$

and is read *the probability of A given B*. The conditional probability of A given B is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Problems.

1. Suppose that each of three persons tosses a coin. If the outcome of one of those tosses differs from the other outcomes, then the game ends. If not, the persons start over and retoss their coins.
   (a) Assuming fair coins, what is the probability that the game will end with the first round of tosses?
   (b) If all three coins are biased and have a probability of $\frac{1}{4}$ of landing on heads, what is the probability that the game will end at the first round?
2. Suppose that a coin that has probability 0.7 of landing on heads is tossed three times. Let X denote the number of heads that appear in the three tosses. Determine the probability distribution of X.

3. **([MM] Example 7.2)** In a simple game of chance, two dice are rolled and the bank pays the player the number of dollars shown on the dice. How much would you pay to play this game?

4. **([MM] 7.4 #7)** The Michigan state lottery runs a game in which you pay $1 to buy a ticket containing a three-digit number of your choice. If your number is drawn at the end of the day, you win $500.

   (a) Suppose you were to buy one ticker per week for a year. What are your chances of coming out a winner for the year? You'll need to decide what is meant by the phrase *coming out a winner*.

   (b) Can you improve your chances of coming out a winner by purchasing more than one ticker per week? Calculate the probability of coming out a winner if you buy n tickets per week for $n = 1, 2, 3, \ldots, 9$.

5. **The geometric distribution.** Consider a trial which has two possible outcomes. We'll call the outcomes *success* and *failure*. As an example, consider flipping a coin as a trial. There are two possible outcomes; namely, heads and tails. Suppose that a success occurs with probability p. Let X denote the random variable that represents the number of independent trials needed to observe a success.

   (a) Find $\mathbb{P}(X = k)$, the probability that a success occurs *for the first time* on the k-th trial.

   (b) Show that $\mathbb{P}(X > k) = (1 - p)^k$. Hint: use the geometric series.

   (c) Show that $\mathbb{P}(X > i + j | X > j) = \mathbb{P}(X > i)$. Interpret this property. Hint for the interpretation: this is often called the memoryless property.

   (d) Compute $\mathbb{E}[X]$. Hint: differentiate the geometric series.

   (e) Suppose that customers arrive at a public telephone at random at the rate of one every 10 minutes. If X is the number of minutes until the next arrival, use the geometric distribution model to compute $P(X > 5)$. For this problem, you'll need to decide what the trials are, what a success means, and what the probability p is.

6. **([MM] Example 7.1)** An electronics manufacturer produces a variety of diodes. Quality control engineers attempt to ensure that faulty diodes will be detected in the factory before they are shipped. It is estimated that 0.3% of the diodes produced will be faulty. It is possible to test each diode individually. It is also possible to place a number of diodes in series and test the entire group. If this test fails, it means that one or more of the diodes in that group are faulty. The estimated testing cost is 5 cents for a single diode, and $4+n$ cents for a group of $n > 1$ diodes. If a group test fails, then each diode in the group must be retested individually to find the bad one(s). Find the most cost-effective quality control procedure for detecting bad diodes. Discuss the sensitivity of your result to the 0.3% assumption.