Math 347
Discrete Dynamical Systems Homework 3

Due Thursday, November 1, 2007

A model of an arms race. We will develop a model of a two-nation arms race. A variation of this model was originally developed by Lewis Fry Richardson in the 1930s. Assume that we have two countries, A and B. Let $A(n)$ and $B(n)$ represent the amount (in some common monetary unit) spent on armaments by the corresponding countries in year n. Assume that each country has some fixed amount of *distrust* of the other country, causing it to retain arms. We will now develop equations that relate the amount each country spends on arms in one year in terms of what they *both* spent the previous year.

First, let's look at the *increase* in expenditures by country A, i.e. $A(n+1)-A(n)$. We wish to construct an equation that takes into account the following information:

- If B spends a lot on defense in one year, then A spends more on defense in the next year.
- Since large expenditures will deplete a country's treasury, large expenditures by A one year will cause smaller expenditures the next year.
- There should be a constant component (i.e. a constant independent of $A(n)$ and $B(n)$) in our expression for $A(n+1)-A(n)$ to account for inflation.

1. Construct an expression for $A(n+1)-A(n)$ that models the scenario described above, and explain the meaning of any constants or parameters that you use.

2. Make similar assumptions about country B to obtain an expression for $B(n+1)-B(n)$:

We now have a dynamical system of two equations, which we will learn how to deal with next week.
For now, we will make some simplifying assumptions.

- Assume that the two countries have an equal amount of distrust of each other.
- Assume that the two countries' economies are about the same.

1. Using the simplifying assumptions above, rewrite your expressions for $A(n+1) - A(n)$ and $B(n+1) - B(n)$.
2. Let $T(n)$ denote the total expenditures on armaments of the two countries, i.e. $T(n) = A(n) + B(n)$. Find an expression for $T(n+1) - T(n)$, and use it to construct a discrete dynamical system model for $T(n)$.
3. Find the equilibrium value for this dynamical system, and discuss its stability.
4. Find a closed-form solution for $T(n)$ in terms of n.
5. Discuss (both mathematically and in terms of the arms race) various possible outcomes (corresponding to various initial conditions $T(0)$ and various relationships between your constants).

Example. Before World War I, there were two alliances: France-Russia and Germany-Austria-Hungary. The estimated total expenditures of these two alliances were (in millions of pounds of sterling): 199 in 1909, 205 in 1910, and 215 in 1911. Thus, letting 1909 be year 0, we have $T(0) = 199$, $T(1) = 205$, and $T(2) = 215$. Use these values to determine the constants (or as many constants as possible) in your discrete dynamical system. Simulate the system numerically and/or graphically (or graph the closed form solution), and discuss the results. Interpret your results.