Math 112: Calculus B
Friday, February 22, 2008
Introduction to Differential Equations

Definition: A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

Some **examples** of differential equations:

Example 1: $y' = xy$ is a first-order differential equation.
Example 2: $y'' + 2y = \sin x$ is a second-order differential equation.
Example 3: $y\frac{dy}{dt} - 2t = y^2$ is a first-order differential equation.
Example 4: $\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$ is a first-order differential equation.
Definition: A **solution** of a differential equation is a function f that satisfies the differential equation when $y = f(x)$ and its derivatives are substituted into the equation. Note that

Some **examples** of solutions of differential equations:

Example 1: Find all solutions of the differential equation

$$y' = 2x + 3.$$

Example 2: Find all solutions of the differential equation

$$y' = y.$$

Example 3: Show that, for any constant C,

$$y(x) = \sqrt{x + C}$$

is a solution of the differential equation

$$y' = \frac{1}{2y}.$$

Example 4: Determine whether $y = x^2$ a solution of the differential equation

$$y'' - 4xy' + 4y = 0.$$

Definition: An **initial-value problem** consists of a differential equation together with a particular initial condition $y(x_0) = y_0$.

Some **examples** of solutions of initial-value problems:

Example 1: Find the unique solution of the initial-value problem

$$y' = 2x + 3, \quad y(0) = 1.$$

Example 2: Find the unique solution of the initial-value problem

$$y' = y, \quad y(0) = -4.$$

Constructing a differential equation.

Example 1: One model for the growth of a population is based on the assumption that the population grows at a rate *proportional to* the size of the population. This is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease). Let $y(t)$ denote the population at time t. Write a differential equation that describes this model. This is called the *natural* or *exponential* growth model.

Example 2: The von Bertalanffy growth model is used to predict the length $L(t)$ of a fish at time t. Let M denote the largest observed length for a species of fish. The model predicts that the rate of change of growth in length *is proportional to* $M - L$, the difference between the current length of the fish and the maximum length. Write a differential equation that describes this model.

Example 3: One model for the spread of an epidemic is that the rate of spread is *jointly proportional to* the number of infected people and the number of uninfected people. Let $y(t)$ denote the number of infected people at time t, and let P denote the total population of a given town. Write a differential equation to model the spread of the epidemic.