Math 347
Tuesday, November 6, 2007
Higher Order Discrete Dynamical Systems

These problems are due on Tuesday, November 13.

1. We have shown previously that $A(n) = Cr^n$ is the general solution for the first-order linear system $A(n+1) - rA(n) = 0$, where C is a constant that can be determined using, for example, initial conditions. In this problem, you will derive a solution for higher-order linear systems. Consider the m-th order discrete dynamical system

$$r_0A(n) + r_1A(n-1) + r_2A(n-2) + \ldots + r_mA(n-m) = 0. \qquad (1)$$

   (a) Assume a solution of the form $A(n) = C\lambda^n$, where C is a constant. Substitute $A(n) = C\lambda^n$ into Eqn. (1).

   (b) Factor out the common factors of C and λ. Since we are not interested in the cases $C = 0$ and $\lambda = 0$ (since these give a zero constant function for $A(n)$), we set the rest equal to zero. The result is called the characteristic equation. Write down the characteristic equation.

   (c) Next, we solve the characteristic equation for λ. What is the degree of the characteristic equation? How many roots will it have (assuming, of course, that some may be complex)? Assume that there are no multiple roots.

   (d) Show that each equation of the form $A(n) = C\lambda^n$, where C is a constant and λ is a root of the characteristic polynomial, is a solution of the original system given by Eqn. (1).

   (e) Show that all linear combinations of these solutions are also solutions of Eqn. (1), and conclude that the general solution of the discrete dynamical system given by Eqn. (1) is

$$A(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_m\lambda_m^n,$$

   where the C_i are constants that can be found provided enough initial information and the λ_i are roots of the characteristic polynomial.

2. **Great Lakes Pollution.** Most of the water flowing into Lake Ontario is from Lake Erie. Here, we will assume that all of the water in Lake Ontario is from Lake Erie and all of the water that leaves Lake Erie goes into Lake Ontario (i.e. we will ignore feeder and outlet streams, evaporation, and other irregularities). Let $a(n)$ and $b(n)$ represent the total amount of pollution in Lake Ontario and Lake Erie, respectively, in year n. Suppose that pollution of these lakes has ceased suddenly. Our goal is to determine how long it would take for the pollution level in each lake to be reduced to 10 percent of its present level. Since the pollution has stopped, the concentration of pollution in the water coming

into Lake Erie is 0. It has been determined that, each year, the percentage of the water replaced in Lakes Erie and Ontario is approximately 38 and 13 percent, respectively. This means that each year, 38 percent of the water in Lake Erie flows into Lake Ontario and is replaced by rain and pure water flowing in from other sources. Also, each year, 13 percent of Lake Ontario's water flows out and is replaced by water flowing in from Lake Erie. Assume that the concentration of the pollution in each lake is constant throughout the lake. We are not interested here in the actual levels of pollution, only in the decrease, so assume that $a(0) = 1$ and $b(0) = 1$.

(a) Write a discrete dynamical system of two equations to model this situation.

(b) Using your equations, derive a single second-order discrete dynamical system model for the amount of pollution in Lake Ontario.

(c) Find the closed-form solution for this equation using the method of the characteristic polynomial discussed in the previous problem. Remember that you know $a(0)$ and $a(1)$, so you can find the necessary constants to obtain the particular solution.

(d) Compute $a(n)$ (for several consecutive values of n) using both the recursive model and the closed-form solution, and verify that you get the same results. Use your simulations to determine how long will it take for Lake Ontario's pollution level to drop to 10% of it's initial level.

(e) Find a closed-form solution for $b(n)$. Use this to determine analytically how long it will take for Lake Erie's to decline to 10% of it's initial level.