Some Facts and Warm-up Questions

- The sum principle: $|A \cup B| = |A| + |B|
- The product principle: $|A \times B| = |A| \cdot |B|
- Show that the number of subsets of an $n$-element set is $2^n$ using the product principle (a version of it).
- Let $A$ be a $k$-element set and $B$ be an $n$-element set. Find the total number of functions from $A$ to $B$, and find the number of one-to-one functions from $A$ to $B$.
- The bijection principle: Two sets have the same size $\iff$ there is a bijection between them.
- Let $\binom{n}{k}$ (read "$n$ choose $k$", and also called "binomial coefficient") denote the number of $k$-element subsets of an $n$-element set. Use a counting argument to show that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Give a counting argument to prove the Binomial Theorem: $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$
- Let $p$ be a prime. Show that $p | \binom{p^i}{i}$ for $1 < i < p$.

1. The powers of 2 are all even and the powers of 5 all end in 5. Show that for any other prime $p$, among the first six powers of $p$ one of these numbers must end in 1.
2. Use the bijection principle to show that $\binom{n}{k} = \binom{n}{n-k}$.
3. Use a counting argument to prove the Pascal’s relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
4. Prove the formula $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ in two ways: using a counting argument, and algebraically using the binomial theorem.
5. Consider the identity $(x + 1)^m(x + 1)^n = (x + 2)^{n+m}$. What identity on binomial coefficients do you get by writing down the corresponding binomial identity. Then, give a counting argument for the identity you obtained.
6. Prove the formula $n \binom{n-1}{k-1} = k \binom{n}{k}$ combinatorially.
7. Prove the formula $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}$ using combinatorial argument.
8. Prove the formula $\sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k}$ (of course not algebraically, but combinatorially)
9. There are 10 people in a room, none of whom is older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?
10. Prove the formula $\binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}$
11. Let $1 \leq t \leq k \leq n$ and $N = \{1, 2, \ldots, n\}$. Classify the $k$-subsets of $N$ by their $t^{th}$ largest element to obtain an identity on binomial coefficients.
12. Give combinatorial arguments to prove each of the following identities

i) \( \binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} \)

ii) \( \sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n \) (Hint: Consider all ternary vectors (vectors whose entries are 0,1 or 2) of length \( n \). Classify them according to a suitable criterion.)

13. Evaluate the sum \( \sum_{k=2}^{n} k(k-2) \binom{n}{k} \)

14. (Putnam 87) Let \( r, s, t \) be integers with \( 0 \leq r, s \) and \( r + s \leq t \). Prove that \( \sum_{i=0}^{s} \binom{s}{i} \binom{t}{r+i} = \frac{t+1}{(t+1-s)(t-s)} \)

15. (Putnam 91) Let \( p \) be an odd prime. Show that \( \sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \mod p^2 \)

(Hint: \( (2 + x)^p(1 + x)^p = ((1 + x) + 1)^p(1 + x)^p) \)