Basic Counting Methods, Binomial Coefficients and Identities

Some Facts and Warm-up Questions

- The sum principle: $|A\dot{\cup}B| = |A| + |B|$
- The product principle: $|A \times B| = |A| \cdot |B|$
- Show that the number subsets of an n-element set is 2^n using the product principle (a version of it).
- Let A be a k-element set and B be an n-element set. Find the total number of functions from A to B, and find the number of one-to-one functions from A to B.
- The bijection principle: Two sets have the same size $\Leftrightarrow$ there is a bijection between them.
- Let $\binom{n}{k}$ (read "n choose k, and also called "binomial coefficient") denote number of k-element subsets of an n-element set. Use a counting argument to show that $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Give a counting argument to prove the Binomial Theorem: $\displaystyle(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$
- Let p be a prime. Show that $p | \binom{p}{i}$ for $1 < i < p$.

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1. The powers of 2 are all even and the powers of 5 all end in 5. Show that for any other prime p, among the first six powers of p one of these numbers must end in 1.
2. Use the bijection principle to show that $\binom{n}{k} = \binom{n}{n-k}$.
3. Use a counting argument to prove the *Pascal's relation* $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
4. Prove the formula $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$ in two ways: using a counting argument, and algebraically using the binomial theorem.
5. Consider the identity $(x+1)^m(x+1)^n = (x+1)^{n+m}$. What identity on binomial coefficients do you get by writing down the corresponding binomial identity. Then, give a counting argument for the identity you obtained.
6. Prove the formula $\displaystyle n\binom{n-1}{k-1} = k\binom{n}{k}$ combinatorially.
7. Prove the formula $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ using combinatorial argument.
8. Prove the formula $\displaystyle\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$ (of course not algebraically, but combinatorially)
9. There are 10 people in a room, none of whom is older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?
10. Prove the formula $\displaystyle\binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}$
11. Let $1 \le t \le k \le n$ and $N = \{1, 2, \ldots, n\}$. Classify the k-subsets of N by their t^{th} largest element to obtain an identity on binomial coefficients.

12. Give combinatorial arguments to prove each of the following identities

i) $\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$

ii) $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$ (Hint: Consider all ternary vectors (vectors whose entries are 0,1 or 2) of length n. Classify them according to a suitable criterion.)

13. Evaluate the sum $\sum_{k=2}^{n} k(k-2)\binom{n}{k}$

14. (Putnam 87) Let r, s, t be integers with $0 \leq r, s$ and $r + s \leq t$. Prove that $\sum_{i=0}^{s} \frac{\binom{s}{i}}{\binom{t}{r+i}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}$

15. (Putnam 91) Let p be an odd prime. Show that $\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \mod p^2$

(Hint: $(2+x)^p(1+x)^p = ((1+x)+1)^p(1+x)^p)$)