Math 347
(Review of) Continuous Probability Models

A continuous random variable X is a variable that can assume an uncountable number of values (corresponding to points on a line interval, for example). Examples of a continuous random variable include the weight of a particular animal species, the length of time it takes to answer a phone at an office, or the height of trees in a forest.

The **probability distribution** associated with a **continuous random variable** X is given by $F(x)$:

$$F(x) = \mathbb{P}(X \leq x).$$

The **probability density function** associated with a continuous random variable $X \in D$ is denoted $f(x) = F'(x)$. Typically, $D = (-\infty, \infty)$ or $D = (0, \infty)$. The probability density function must satisfy:

- $\int_D f(x) = 1$.

- $f(x) \geq 0$ for all $x \in D$.

The normal and exponential probability distributions are among the most commonly used continuous probability distributions in mathematical modeling.

The mean, or average, or **expected value** of X is given by

$$\mathbb{E}[X] = \int_D x f(x)\, dx.$$

The **variance** of X measures the extent to which X deviates from the mean, and is given by

$$\mathrm{Var}[X] = \int_D (x - \mathbb{E}[X])^2 f(x)\, dx.$$

The probability that $a \leq X \leq b$ is given by

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

As before, two *events* A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

The probability of A or B is

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

Note that if events A and B are mutually exclusive, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

Conditional probability is the probability that some event, say A, occurs, given that some other event, say B, has already occurred. Conditional probability is written

$$\mathbb{P}(A|B),$$

and is read *the probability of A given B*. The conditional probability of A given B is defined by

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Problems.

1. Derive the mean and variance for the continuous random variable X, where X is the random variable with density function

$$f(x) = 4x^2 e^{-2x}, \ D = (0, \infty).$$

This density function is used in quantum chemistry to describe the location of an electron around the nucleus in a hydrogen atom.

2. Derive the mean and variance for the continuous random variable X, where X is *uniform* over the interval $[a, b]$, where $a < b$. This means that the probability density function of X is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

3. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1 - x^2) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of c?

(b) What is the distribution function of X?

4. Derive the mean and variance for the continuous random variable X, where X is exponential with parameter λ. This means that the probability density function of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

What is the probability distribution function of X?

5. Suppose that X is exponential with parameter λ. Find

$$\mathbb{P}(X > s + t | X > s).$$

   The exponential distribution is often used to model arrival processes (e.g. arrivals of customers, phone calls, radioactive decays), where X denotes the time between two successive arrivals. Interpret the property that you have just derived in terms of this application. Hint: the property is often called the *memoryless property* of the exponential distribution.

6. Suppose that the amount of time that one spends in a bank is exponentially distributed with mean ten minutes. What is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that he is still in the bank after ten minutes? Hint: you'll need to use the mean to find the parameter λ.

7. Suppose that X and Y are independent exponential random variables with parameters λ and μ, respectively. Define a new random variable W by

$$W = \min(X, Y).$$

   Show that W is an exponential random variable with parameter $\lambda + \mu$. Hint: compute $\mathbb{P}(W > s)$, and identify the density and distribution functions of W.

8. Consider a stereo system consisting of two main components, a receiver and a speaker. Suppose that the lifetime of the receiver is exponential with mean 1000 hours and that the lifetime of the speaker is exponential with mean 500 hours, independent of the lifetime of the receiver. Find the probability that the stereo system's failure (when it occurs) will be caused by the receiver failing to function.

9. Consider a bank that has two clerks. Suppose that service times at this bank are independent and exponentially distributed with parameter λ. When the bank opens at 9:00 am, you enter the bank together with two other customers. You are generous and let the other two customers proceed to the two clerks. You will then be the next to be served by the next available clerk. What is the probability that, of the three customers, you will be the last to leave?