Math 333
Complex Roots of the Characteristic Equation

Let's return to the second order linear homogeneous differential equation with constant coefficients:

$$ay'' + by' + cy = 0. \tag{1}$$

Recall that if the roots r_1 and r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are real and different, then the general solution of Eqn. (1) is

$$y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t}.$$

Suppose now that the characteristic equation

$$ar^2 + br + c = 0$$

has two complex conjugate roots:

$$r_1 = \lambda + i\mu \text{ and } r_2 = \lambda - i\mu,$$

where λ and μ are real. Then

$$y_1(t) = e^{(\lambda + i\mu)t}$$

and

$$y_2(t) = e^{(\lambda - i\mu)t}$$

are two solutions of Eqn. (1).

Unfortunately, the solutions y_1 and y_2 are complex-valued functions, and in general we would like to have real-valued solutions, if possible, since the original differential equation has real coefficients. To find real-valued solutions, first recall Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using Euler's formula, we can rewrite the solutions y_1 and y_2 as

$$y_1(t) = e^{\lambda t}e^{i\mu t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$$

and

$$y_2(t) = e^{\lambda t}e^{-i\mu t} = e^{\lambda t}(\cos(\mu t) - i\sin(\mu t)).$$

Moreover, recall from the **Linearity Principle** that if y_1 and y_2 are solutions of Eqn. (1), then any linear combination of y_1 and y_2 is also a solution. In particular, let's form the sum and then the difference of y_1 and y_2. We have:

$$y_1(t) + y_2(t) = 2e^{\lambda t}\cos(\mu t)$$

and

$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin(\mu t).$$

Next, neglecting the constant multipliers 2 and $2i$, respectively, we have obtained a pair of real-valued solutions

$$u(t) = e^{\lambda t}\cos(\mu t) \text{ and } v(t) = e^{\lambda t}\sin(\mu t).$$

You should verify that $u(t)$ and $v(t)$ are indeed solutions of Eqn. (1). Note that $u(t)$ and $v(t)$ are simply the real and imaginary parts, respectively, of y_1.

Next, we'd like to use $u(t)$ and $v(t)$ to construct the *general solution* of Eqn. (1). To do this, show that the Wronskian of u and v is

$$W(u(t), v(t)) = \mu e^{2\lambda t}.$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. Of course, if $\mu = 0$, then the roots of the characteristic equation are real, so the previous discussion does not apply. Thus, $W(u, v) \neq 0$, so we conclude that $u(t)$ and $v(t)$ indeed form a fundamental set of solutions for Eqn. (1). Finally, we conclude that the general solution of Eqn. (1) is

$$y(t) = k_1 u(t) + k_2 v(t).$$

Summary. Suppose that the roots of the characteristic equation are complex numbers $\lambda \pm i\mu$ with $\mu \neq 0$. Then the general solution of Eqn. (1) is

$$y(t) = k_1 e^{\lambda t}\cos(\mu t) + k_2 e^{\lambda t}\sin(\mu t),$$

where k_1 and k_2 are arbitrary constants.

Examples.

1. Find the general solution of

$$y'' + y' + y = 0.$$

Solution. $y = k_1 e^{-t/2}\cos(\sqrt{3}t/2) + k_2 e^{-t/2}\sin(\sqrt{3}t/2)$

2. Find the general solution of

$$y'' + 9y = 0.$$

Solution. $y = k_1\cos(3t) + k_2\sin(3t)$.

3. Find the solution of the initial-value problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

Solution. $y = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t)$