§11.8 Chapter Summary

1. $\lim_{k\to\infty} a_k = \infty$.

2. $\lim_{k\to\infty} a_k = 0$.

3. $\lim_{k\to\infty} a_k = \infty$.

4. $\lim_{k\to\infty} a_k = \infty$.

5. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{n^3+4} = \lim_{n\to\infty} \frac{1}{3n^2} = 0$.

6. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln(1+k^2)}{\ln(4+k)} = \lim_{k\to\infty} \frac{2k(4+k)}{1+k^2} = 2$.

7. $\lim_{k\to\infty} a_k = 0$.

8. $\lim_{k\to\infty} a_k = 0$.

9. Converges absolutely — $\sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^2 \le \sum_{k=0}^{\infty} \frac{1}{k!} = e$.

10. Converges absolutely. $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3} \le \frac{1}{3(\ln 3)^3} + \int_3^{\infty} \frac{dx}{x(\ln x)^3} = \frac{1}{3(\ln 3)^3} + \frac{1}{2(\ln 3)^2}$.

11. Diverges by the nth term test: $\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} = \infty$.

12. Diverges. The partial sum of the first $2n$ terms of the series is one-half the partial sum of the first n terms of the harmonic series.

13. Converges—by the integral test: $\sum_{j=1}^{\infty} j \cdot 5^{-j} \le \frac{1}{5} + \int_1^{\infty} x \cdot 5^{-x}\, dx = \frac{1}{5} + \frac{1+\ln 5}{5(\ln 5)^2}$.

    NOTE: This series can also be shown to converge via the ratio test:

$$\lim_{j\to\infty} \frac{\frac{j+1}{5^{j+1}}}{\frac{j}{5^j}} = \lim_{j\to\infty} \frac{j+1}{j} \cdot \frac{5^j}{5^{j+1}} = \lim_{j\to\infty} \frac{j+1}{5j} = \frac{1}{5} < 1.$$

    Furthermore, since $a_{j+1}/a_j = (j+1)/5j \le 2/5$ when $j \ge 1$, $\sum_{j=1}^{\infty} j \cdot 5^{-j} \le \sum_{j=1}^{\infty} \frac{2^{j-1}}{5^j} = \frac{1}{3}$.

14. Converges—by the comparison test: $\sum_{j=1}^{\infty} \frac{j}{j^4+j-1} < \sum_{j=1}^{\infty} \frac{j}{j^4} = \sum_{j=1}^{\infty} \frac{1}{j^3} \le \frac{3}{2}$.

15. Converges—by the integral test: $\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$.
$\sum_{m=0}^{\infty} e^{-m^2} \le 1 + \int_0^{\infty} e^{-x^2}\,dx = 1 + \sqrt{\pi}/2$.

16. Diverges—by the comparison test:
$\sum_{m=1}^{\infty} \frac{m^3}{m^4-7} = -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{m^3}{m^4-7} > -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{m^3}{m^4} = -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{1}{m}$.

17. Diverges—by the comparison test: $\sum_{k=1}^{\infty} \frac{k!}{(k+1)!-1} > \sum_{k=1}^{\infty} \frac{k!}{(k+1)!} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \sum_{k=2}^{\infty} \frac{1}{k}$.

18. Converges—by the integral test: $\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \int_{\ln 2}^{\infty} ue^{-u}\,du = \frac{1+\ln 2}{2}$. Thus,
$\sum_{j=2}^{\infty} \frac{\ln j}{j^2} \le \frac{\ln 2}{4} + \frac{1+\ln 2}{2} = \frac{2+3\ln 2}{4}$.

19. Converges—by the comparison test: $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \le 1 + \int_1^{\infty} x^{-3/2}\,dx = 3$.

20. The series converges absolutely by the comparison test using $b_k = (2/7)^k$.
$\left| S - \sum_{k=0}^{N} a_k \right| \le 0.005$ when $N \ge 4$ since $2^5/(7^5+5) < 0.005$. Using $N = 4$,
$S \approx \frac{68917177}{84877260} \approx 0.81196$.

21. The series converges absolutely by the comparison test: $\sum_{k=0}^{\infty} \frac{1}{(k+1)\,2^k} < \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.
$\left| S - \sum_{k=0}^{N} a_k \right| \le 0.005$ when $N \ge 5$ since $1/(7 \cdot 2^6) = 1/448 < 0.005$. Using $N = 5$,
$S \approx \frac{259}{320} = 0.809375$.

22. converges absolutely—$\sum_{m=8}^{\infty} \left| \frac{\sin m}{m^3} \right| < \sum_{m=8}^{\infty} \frac{1}{m^3}$. Since
$\left| \sum_{m=8}^{\infty} \frac{\sin m}{m^3} \right| \le \sum_{m=8}^{\infty} \frac{1}{m^3} \le \frac{1}{8^3} + \int_8^{\infty} \frac{dx}{x^3} = \frac{5}{512}$, it follows that $-\frac{5}{512} \le \sum_{m=8}^{\infty} \frac{\sin m}{m^3} \le \frac{5}{512}$.

23. converges conditionally—$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is (almost) the alternating harmonic series.

$$-1 < \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -\ln 2 < -\frac{1}{2}.$$

24. Diverges — $\lim_{k\to\infty} \frac{3^k}{k^3+3^k} \neq 0$.

25. Diverges by the nth term test. (Since $0 < \pi - e$, $\lim_{k\to\infty} k^{\pi-e} = \infty$.)

26. $\sum_{m=2}^{\infty} \frac{1}{(\ln 3)^m} = \frac{1}{(\ln 3)^2} \frac{1}{1-(1/\ln 3)} = \frac{1}{(\ln 3)(\ln 3 - 1)}$.

27.
$$\sum_{j=0}^{\infty}\left(\frac{1}{2^j}+\frac{1}{3^j}\right)^2 = \sum_{j=0}^{\infty}\left(\frac{1}{2^{2j}}+\frac{2}{6^j}+\frac{1}{3^{2j}}\right) = \sum_{j=0}^{\infty}\left(\frac{1}{4^j}+\frac{2}{6^j}+\frac{1}{9^j}\right) = \frac{4}{3}+\frac{12}{5}+\frac{9}{8} = \frac{583}{120}.$$

28. $\sum_{k=1}^{\infty}\left(\int_k^{k+1}\frac{dx}{x^2}\right) = \sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$.

29. Diverges by the nth term test: $\lim_{m\to\infty}\int_0^m e^{-x^2}\,dx = \sqrt{\pi}/2 \neq 0$.

30. Diverges by the nth term test: $\lim_{n\to\infty}\frac{\ln n}{\ln(3+n^2)} = \frac{1}{2} \neq 0$.

31. **No.** Using the series representation of $\sin x$ and the alternating series test, $\sin(1/n) > 1/n - 1/6n^3 = (6n^2-1)/6n^3 \geq 5/6n$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty}\sin(1/n) > \frac{5}{6}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$.

32. **Yes.** Since $0 < \sin(1/n) < 1/n$ for all $n \geq 1$, $0 < \sum_{n=1}^{\infty}\frac{1}{n}\sin(1/n) < \sum_{n=1}^{\infty}\frac{1}{n^2}$. Thus, the given series converges by the comparison test.

33. **No.** Since $\lim_{n\to\infty} e^{-1/n} = 1 \neq 0$, the series diverges by the n-th term test.

34. **No.** Using the power series representation of e^x and the alternating series theorem,

$$1 - e^{-1/n} > \frac{1}{n} - \frac{1}{2n^2} = \frac{2n-1}{2n^2} \geq \frac{2n-n}{2n^2} = \frac{1}{2n}$$

for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}\left(1-e^{-1/n}\right) \geq \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$.

35. (a) $\ln(n!) = \ln n + \ln(n-1) + \ln(n-2) + \cdots + \ln 2 + \ln 1$. Thus, (since $\ln 1 = 0$) $\ln(n!)$ is a right sum approximation so $\int_1^n \ln x\,dx$. Since $\ln x$ is an increasing function on the interval $[1, n]$, $\ln(n!) > \int_1^n \ln x\,dx = n\ln n - n + 1$. Therefore, $n! > e^{n\ln n - n + 1} = n^n e^{1-n}$.

(b) Part (a) implies that $\dfrac{b^N}{N!} < \left(\dfrac{be}{N}\right)^N \dfrac{1}{e}$. Thus, $\dfrac{b^N}{N!} < \dfrac{1}{2}$ when $\left(\dfrac{be}{N}\right)^N < \dfrac{e}{2}$. Therefore, since $\dfrac{e}{2} > 1$, any $N > be$ satisfies the given inequality.

36. (a) For all $k \geq 1$, $0 < a_k < 1/k \implies \lim_{k\to\infty} a_k = 0$.

(b) **No**, the series diverges. Since $a_k = \displaystyle\int_k^\infty \frac{dx}{2x^2 - 1} \geq \int_k^\infty \frac{dx}{2x^2} = \frac{1}{2k}$,
$$\sum_{k=1}^\infty a_k \geq \frac{1}{2}\sum_{k=1}^\infty \frac{1}{k} = \infty.$$

(c) $\displaystyle\sum_{k=1}^\infty (-1)^{k+1} a_k$ converges by the alternating series test — the terms of the series alternate in sign and are decreasing in magnitude (i.e., $a_{k+1} < a_k$).

37. $\left|\dfrac{a_{m+1}}{a_m}\right| = \dfrac{m^2+1}{(m+1)^2+1} \cdot |x| < 1 \implies R = 1$. The interval of convergence is $[-1, 1]$.

38. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left(\dfrac{n+1}{n}\right)^n \cdot (n+1) \cdot |x| < 1 \implies R = 0$. The interval of convergence is just $x = 0$.

39. The interval of convergence is $(-1/3, 1/3)$.

40. The interval of convergence is $(-\infty, \infty)$.

41. The interval of convergence is $[-1/3, 1/3)$.

42. The interval of convergence is $[-1/3, 1/3]$.

43. $(1, 5)$.

44. $(-\infty, \infty)$.

45. $[-3, 5)$.

46. $[0, 2)$.

47. $[-6, -4]$.

48. $[1/2, 3/2)$.

49. **Cannot** be true. The interval of convergence of a power series is symmetric around and includes its base point ($b = 1$ in this case).

50. **May** be true. (The statement is true when $a_k = 1/k!$ but it is false when $a_k = 1$.)

51. **Must** be true. If the radius of convergence of the power series is 3, then the interval of convergence includes all values of x such that $|x - 1| < 3$.

52. **Cannot** be true. The interval of convergence of this power series must be symmetric about the point $b = 1$.

53. **Cannot** be true. The interval of convergence of the power series is the solution set of the inequality $|x - 1| < 8$. Thus, the radius of convergence of the power series is 8.

54. (a) $\lim_{x\to1^-} \sum_{k=0}^{\infty} (-1)^k x^k = \lim_{x\to1^-} \frac{1}{1+x} = \frac{1}{2}$.

(b) An infinite series converges only if the sequence defined by its partial sums converges. Since the partial sums $\sum_{k=0}^{N} (-1)^k$ are alternately 1 and 0, the infinite series does not converge.

55. $\frac{1-\cos x}{x} = x^{-1}\left(1 - \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}\right) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{(2k)!} \implies \lim_{x\to0}\frac{1-\cos x}{x} = 0.$

56. $\frac{e^x - e^{-x}}{x} = 2\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k+1)!} \implies \lim_{x\to0}\frac{e^x - e^{-x}}{x} = 2.$

57. $\frac{x - \arctan x}{x^3} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2k+3} \implies \lim_{x\to0}\frac{x-\arctan x}{x^3} = \frac{1}{3}.$

58. $\lim_{x\to1}\frac{\ln x}{x-1} = \lim_{w\to0}\frac{\ln(1+w)}{w} = \lim_{w\to0}\left(\sum_{k=0}^{\infty}(-1)^k\frac{w^k}{k+1}\right) = 1.$

59. $2^x = e^{x\ln 2} = \sum_{k=0}^{\infty}\frac{(x\ln 2)^k}{k!}$; $R = \infty$.

60. $\cos^2 x = \frac{1}{2}(1+\cos(2x)) = \frac{1}{2}\left(1 + \sum_{k=0}^{\infty}(-1)^k\frac{(2x)^{2k}}{(2k)!}\right) = \frac{1}{2} + \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k-1}x^{2k}}{(2k)!}$; $R = \infty$.

61.
$$\begin{aligned}\frac{5+x}{x^2+x-2} &= \frac{2}{x-1} - \frac{1}{x+2} = -\frac{2}{1-x} - \frac{1}{2}\frac{1}{1+(x/2)}\\ &= -2\sum_{k=0}^{\infty}x^k - \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^k\\ &= -\sum_{k=0}^{\infty}\left(\frac{2^{k+2}+(-1)^k}{2^{k+1}}\right)x^k; \quad R = 1.\end{aligned}$$

62. $f(x) = \sin^3(x) = \frac{1}{4}\big(3\sin x - \sin(3x)\big) = \frac{3}{4}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} - \frac{1}{4}\sum_{k=0}^{\infty}(-1)^k\frac{(3x)^{2k+1}}{(2k+1)!} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^{2k+1}-3}{4\cdot(2k+1)!}x^{2k+1}$; $R = \infty$.

63. $\dfrac{1}{1-x} = -\dfrac{1}{1+(x-2)} = -\sum_{k=0}^{\infty}(-1)^k(x-2)^k \implies a_k = (-1)^{k+1}$.

64. Since $f'(0) > 0$, the coefficient of x in the Maclaurin series representation of f must be positive; the coefficient of x in the series given is negative.

65. (a) **No**. The Maclaurin series representation of f is $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \cdots$. Since f is concave down at $x = 0$, the coefficient of x^2 in the Maclaurin series representation of f is negative.

(b) **Yes**. $g''(0) = \dfrac{3}{4}\big(f'(0)\big)^2\big(g(0)\big)^5 - \dfrac{1}{2}f''(0)\big(g(0)\big)^3 > 0$.

66. (a) The following inequalities are apparent from the figures illustrating the integral test:

$$\int_1^{n+1} a(x)\,dx < \sum_{k=1}^{n} a_k < a_1 + \int_1^n a(x)\,dx.$$

Taking $a(x) = 1/x$, these inequalities imply that $\ln(n+1) < H_n < 1 + \ln n$.

(b) First, observe that $a_n - a_{n+1} = \big(H_n - \ln n\big) - \big(H_{n+1} - \ln(n+1)\big) = \ln(n+1) - \ln n - \dfrac{1}{n+1}$. Then, note that $\int_n^{n+1} x^{-1}\,dx = \ln(n+1) - \ln n > \dfrac{1}{n+1}$ since x^{-1} is a decreasing function on the interval $[n, n+1]$. Thus, $a_n > a_{n+1}$.

(c) The sequence a_n is decreasing and bounded below by 0 (since $H_n - \ln n > \ln(n+1) - \ln n > 0$). Thus, it converges.

(d) $\int_x^{\infty} f'(t)\,dt = \lim_{a\to\infty}\int_x^a f'(t)\,dt = \lim_{a\to\infty}\big(f(a) - f(x)\big) = -f(x)$. [Note: $f(a) = \ln\left(\dfrac{a+1}{a}\right) - \dfrac{1}{a+1} = \ln\left(1 + \dfrac{1}{a}\right) - \dfrac{1}{a+1}$.]

(e) When $x > 0$, $f(x) = -\int_x^{\infty} f'(t)\,dt > \int_x^{\infty}\dfrac{dt}{(t+1)^3} = \dfrac{1}{2(x+1)^2}$.

(f) Let

$$\begin{aligned} S_N &= \sum_{k=n}^{N}(a_k - a_{k+1}) \\ &= (a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \cdots + (a_N - a_{N+1}) = a_n - a_{N+1}. \end{aligned}$$

Since $\gamma = \lim_{n\to\infty} a_n$, $\sum_{k=n}^{\infty}(a_k - a_{k+1}) = \lim_{N\to\infty} S_N = a_n - \lim_{N\to\infty} a_{N+1} = a_n - \gamma$.

(g) Since f is a decreasing function, the integral test implies that $\int_n^\infty f(x)\,dx \le \sum_{k=n}^{\infty} f(k)$.

Therefore, part (b) implies that $\int_n^\infty f(x)\,dx > \dfrac{1}{2(n+1)}$.

To get the upper bound on $a_n - \gamma$, note that $f(k) < \dfrac{1}{2}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)$. (Apply the trapezoid rule to $\int_k^{k+1} dx/x$.) This inequality implies that

$$\sum_{k=n}^{\infty} f(k) < \sum_{k=n}^{\infty} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2n}.$$

67. Since $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{|r-n|\cdot|x|}{n+1} < 1$ if $|x| < 1$, the binomial series converges if $|x| < 1$.

68. Let $r = 1/2$.

69. The binomial series for $(1+u)^3$ terminates after a finite number of terms since $r = 3$ is an integer: $(1+u)^3 = 1 + 3u + 3u^2 + u^3$. Therefore, $f(x) = \left(1+x^4\right)^3 = 1 + 3x^4 + 3x^8 + x^{12}$.

70. $g(x) = \sqrt[3]{1-x^2} \approx 1 - x^2/3 - x^4/9 - 5x^6/81$.

71. $g(x) = (1+x^2)^{-3/2} \approx 1 - 3x^2/2 + 15x^4/8 - 35x^6/16$.

72.
$$g(x) = \arcsin x = \int \frac{dx}{\sqrt{1-x^2}} = \int\left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \cdots\right)dx \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}.$$

73. Since $\sqrt{1+u} = 1 + \dfrac{u}{2} - \dfrac{u^2}{8} + \dfrac{u^3}{16} - \dfrac{5u^4}{128} \pm \cdots$,

$\sqrt{1+x^3} = 1 + \dfrac{x^3}{2} - \dfrac{x^6}{8} + \dfrac{x^9}{16} - \dfrac{5x^{12}}{128} \pm \cdots$. Thus,

$$\int_0^{2/5}\sqrt{1+x^3}\,dx = \int_0^{2/5}\left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} \pm \cdots\right)dx$$
$$= \frac{2}{5} + \frac{1}{8}\left(\frac{2}{5}\right)^4 - \frac{1}{56}\left(\frac{2}{5}\right)^7 + \frac{1}{160}\left(\frac{2}{5}\right)^{10} - \frac{5}{1664}\left(\frac{2}{5}\right)^{13} \pm \cdots.$$

Now, since this is an alternating series (after the first term) and $2^7/(56\cdot 5^7) < 5\times 10^{-4}$, the value of the integral is approximated to the desired accuracy by $2/5 + 2^4/(8\cdot 5^4) = 252/625 = 0.4032$.

74. (a) $f'(x) = \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!} x^{(n-1)}$. The coefficient of x^n in the series for $(1+x)f'(x)$ is

$$\frac{r(r-1)(r-2)\cdots(r-n+1)(r-n)}{n!} + \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!}$$
$$= \frac{(r-n)\cdot r(r-1)(r-2)\cdots(r-n+1)}{n!} + \frac{n \cdot r(r-1)(r-2)\cdots(r-n+1)}{n!}$$
$$= \frac{r \cdot r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

Since this is also the coefficient of x^n in the series for $rf(x)$, the result follows.

(b) $g'(x) = \frac{f'(x)}{(1+x)^r} - \frac{rf(x)}{(1+x)(1+x)^r} = \frac{f'(x)}{(1+x)^r} - \frac{(1+x)f'(x)}{(1+x)(1+x)^r} = 0.$

(c) The result in part (c) implies that g is a constant function. Since $g(0) = 1$, $g(x) = 1$ and so $f(x) = (1+x)^r$.

75. (a) By the binomial power series,

$$f(x) = (1+x)^r = 1 + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n.$$

Thus, $g(x) = (1-x^2)^{-1/2}$ can be written as

$$1 + \sum_{n=1}^{\infty} \frac{(-1/2)(-3/2)(-5/2)\cdots(1/2-n)}{n!}(-x^2)^n$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(-1)(-3)(-5)\cdots(1-2n)}{2^n \cdot n!}x^{2n}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n}$$

To finish, we integrate:

$$\int \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n}\right) dx = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{x^{2n+1}}{2n+1}$$
$$= \int_0^x \frac{dt}{\sqrt{1-t^2}}$$
$$= \arcsin x.$$

(b) Let $u = \arcsin x$. Then $du = \frac{dx}{\sqrt{1-x^2}}$ and $\sin u = x$. Thus,

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}dx = \int_0^{\pi/2} \sin^{2n+1} u \, du$$
$$= \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}.$$

(c) $$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{\arcsin x}{x^{2n+1}} \cdot \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$$
$$= \int_0^1 \left(x^{-2n} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{1}{2n+1} \right) \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$$
$$= \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \left(\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{1}{2n+1} \right) \int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$$
$$= 1 + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

(d) Let $u = \arcsin x$. Then $du = 1/\sqrt{1-x^2}dx$. Thus,

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} u\,du = \frac{\pi^2}{8}$$

(e) After writing out a few terms, it is clear that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.$$

Since the summation on the left hand side is a convergent p-series, we can assign to it some limit L. Then, the above equation can be rewritten as $L = \pi^2/8 + L/4$. Solving for L yields $L = \pi^2/6$. Thus, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

76. (a) By part (c) of the Exercise 19 in Section 9.2, $R_n = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt$. Let $f(t) = (1+t)^r$. After differentiating f a few times, it is easy to see that $f'(t) = r(r-1)(r-2)\cdots(r-n)(1+t)^{r-(n+1)}$. Thus $R_n = \frac{r\cdot(r-1)\cdot(r-2)\cdots(r-n)}{n!}\int_0^x \frac{(x-t)^n}{(1+t)^{n+1-r}}\,dt$.

(b) Since $t \geq 0$, $\frac{|x-t|}{1+t} \leq |x-t| \leq |x|$.

(c) By parts (a) and (b),

$$|R_n(x)| \leq \frac{|r\cdot(r-1)\cdot(r-2)\cdots(r-n)|}{n!} x^n \int_0^x |(1+t)^{r-1}|\,dt$$
$$= \frac{|(r-1)\cdot(r-2)\cdots(r-n)|}{n!} x^n |(1+x)^r - 1|.$$

(d) If $0 \leq x < 1$, then $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|(r-1)\cdot(r-2)\cdots(r-n)|}{n!} x^n |(1+x)^r - 1| = 0$. Thus, the binomial series converges to f along the interval $[0, 1)$.

(e) If $-1 < x \le t < 0$, then $\dfrac{-|x-t|}{1+t} \le |x|$. Thus,

$$|R_n(x)| \le \frac{|r \cdot (r-1) \cdot (r-2) \cdots (r-n)|}{n!} |x|^n \int_0^x |(1+t)^{r-1}|\, dt$$
$$= \frac{|r \cdot (r-1) \cdot (r-2) \cdots (r-n)|}{n!} |x|^n |1-(1+x)^r|.$$

If $-1 < x < 0$, this series converges converges as $n \to \infty$, since $\lim_{n\to\infty} |x|^n = 0$.

77. (a) Notice that $\lim_{m\to\infty} \sum_{k=0}^{m} \dfrac{(-1)^k}{k!} = e^{-1}$. Furthermore, since $\sum_{k=0}^{m} \dfrac{(-1)^k}{k!}$ is an alternating series, $\left| \dfrac{1}{e} - \sum_{k=0}^{m} \dfrac{(-1)^k}{k!} \right| \le \dfrac{1}{(m+1)!}$. Therefore,
$m! \left| \dfrac{1}{e} - \sum_{k=0}^{m} \dfrac{(-1)^k}{k!} \right| \le \dfrac{m!}{(m+1)!} = \dfrac{1}{m+1}$.

(b) $m!/e = (n \cdot m!)/m = n \cdot (m-1)!$. Since this final expression is a product of integers, $m!/e$ is also an integer.

(c) Let $a_k = m!/k!$, where k is an integer such that $0 \le k \le m$. Then, $a_k = m(m-1)(m-2)\cdots(m-(m-k-1))$. Since a_k is a product of integers, it, too, is an integer. Thus, since $m! \sum_{k=0}^{m} \dfrac{(-1)^k}{k!}$ is the sum of alternatingly positive and negative integers, the summation itself is an integer.

(d) By assumption, m is a positive integer. Thus, $1/(m+1) \le 1/2$ for all m. Since N is the product of two non-negative integer values, it follows that $N = 0$.

(e) By the previous part $m! \left| \dfrac{1}{e} - \sum_{k=0}^{m} \dfrac{(-1)^k}{k!} \right| = 0$. Since $m! > 0$, it follows that

$$\frac{1}{e} = \sum_{k=0}^{m} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=m+1}^{\infty} \frac{(-1)^k}{k!}.$$

Thus, $\sum_{k=m+1}^{\infty} \dfrac{(-1)^k}{k!} = 0$. However, this is false, since the terms of the summation are both alternating and decreasing. Therefore, e is irrational.

78. (a) The sum of k numbers is less than or equal to k times the largest summand; similarly, it is greater than or equal to k times the smallest summand. Thus, since the sequence $\{a_n\}$ is decreasing,

$$2^{m-1} a_{2^m} \le a_{2^{m-1}+1} + a_{2^{m-1}+2} + \cdots + a_{2^m} \le 2^{m-1} a_{2^{m-1}+1} \le 2^{m-1} a_{2^{m-1}}.$$