§11.8 Chapter Summary

1. $\lim_{k\to\infty} a_k = \infty$.

2. $\lim_{k\to\infty} a_k = 0$.

3. $\lim_{k\to\infty} a_k = \infty$.

4. $\lim_{k\to\infty} a_k = \infty$.

5. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n+2}{n^3+4} = \lim_{n\to\infty} \dfrac{1}{3n^2} = 0$.

6. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{\ln\left(1+k^2\right)}{\ln(4+k)} = \lim_{k\to\infty} \dfrac{2k(4+k)}{1+k^2} = 2$.

7. $\lim_{k\to\infty} a_k = 0$.

8. $\lim_{k\to\infty} a_k = 0$.

9. Converges absolutely — $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)^2 \le \sum_{k=0}^{\infty} \frac{1}{k!} = e$.

10. Converges absolutely. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^3} \le \frac{1}{3(\ln 3)^3} + \int_3^{\infty} \frac{dx}{x(\ln x)^3} = \frac{1}{3(\ln 3)^3} + \frac{1}{2(\ln 3)^2}$.

11. Diverges by the nth term test: $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} = \infty$.

12. Diverges. The partial sum of the first $2n$ terms of the series is one-half the partial sum of the first n terms of the harmonic series.

13. Converges—by the integral test: $\displaystyle\sum_{j=1}^{\infty} j \cdot 5^{-j} \le \frac{1}{5} + \int_1^{\infty} x \cdot 5^{-x}\, dx = \frac{1}{5} + \frac{1+\ln 5}{5(\ln 5)^2}$.

    NOTE: This series can also be shown to converge via the ratio test:

$$\lim_{j\to\infty} \frac{\frac{j+1}{5^{j+1}}}{\frac{j}{5^j}} = \lim_{j\to\infty} \frac{j+1}{j} \cdot \frac{5^j}{5^{j+1}} = \lim_{j\to\infty} \frac{j+1}{5j} = \frac{1}{5} < 1.$$

    Furthermore, since $a_{j+1}/a_j = (j+1)/5j \le 2/5$ when $j \ge 1$, $\displaystyle\sum_{j=1}^{\infty} j \cdot 5^{-j} \le \sum_{j=1}^{\infty} \frac{2^{j-1}}{5^j} = \frac{1}{3}$.

14. Converges—by the comparison test: $\displaystyle\sum_{j=1}^{\infty} \frac{j}{j^4+j-1} < \sum_{j=1}^{\infty} \frac{j}{j^4} = \sum_{j=1}^{\infty} \frac{1}{j^3} \le \frac{3}{2}$.

15. Converges—by the integral test: $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.
$\sum_{m=0}^{\infty} e^{-m^2} \le 1 + \int_0^\infty e^{-x^2}\,dx = 1 + \sqrt{\pi}/2$.

16. Diverges—by the comparison test:
$$\sum_{m=1}^{\infty} \frac{m^3}{m^4-7} = -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{m^3}{m^4-7} > -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{m^3}{m^4} = -\frac{1}{6} + \sum_{m=2}^{\infty} \frac{1}{m}.$$

17. Diverges—by the comparison test: $\sum_{k=1}^{\infty} \frac{k!}{(k+1)!-1} > \sum_{k=1}^{\infty} \frac{k!}{(k+1)!} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \sum_{k=2}^{\infty} \frac{1}{k}$.

18. Converges—by the integral test: $\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_{\ln 2}^\infty u e^{-u}\,du = \frac{1+\ln 2}{2}$. Thus,
$\sum_{j=2}^{\infty} \frac{\ln j}{j^2} \le \frac{\ln 2}{4} + \frac{1+\ln 2}{2} = \frac{2+3\ln 2}{4}$.

19. Converges—by the comparison test: $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \le 1 + \int_1^\infty x^{-3/2}\,dx = 3$.

20. The series converges absolutely by the comparison test using $b_k = (2/7)^k$.
$\left| S - \sum_{k=0}^{N} a_k \right| \le 0.005$ when $N \ge 4$ since $2^5/(7^5+5) < 0.005$. Using $N = 4$,
$S \approx \frac{68917177}{84877260} \approx 0.81196$.

21. The series converges absolutely by the comparison test: $\sum_{k=0}^{\infty} \frac{1}{(k+1)\,2^k} < \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$.
$\left| S - \sum_{k=0}^{N} a_k \right| \le 0.005$ when $N \ge 5$ since $1/(7 \cdot 2^6) = 1/448 < 0.005$. Using $N = 5$,
$S \approx \frac{259}{320} = 0.809375$.

22. converges absolutely—$\sum_{m=8}^{\infty} \left| \frac{\sin m}{m^3} \right| < \sum_{m=8}^{\infty} \frac{1}{m^3}$. Since
$\left| \sum_{m=8}^{\infty} \frac{\sin m}{m^3} \right| \le \sum_{m=8}^{\infty} \frac{1}{m^3} \le \frac{1}{8^3} + \int_8^\infty \frac{dx}{x^3} = \frac{5}{512}$, it follows that $-\frac{5}{512} \le \sum_{m=8}^{\infty} \frac{\sin m}{m^3} \le \frac{5}{512}$.

23. converges conditionally—$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is (almost) the alternating harmonic series.

$$-1 < \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = -\ln 2 < -\frac{1}{2}.$$

24. Diverges — $\lim_{k\to\infty} \frac{3^k}{k^3+3^k} \neq 0.$

25. Diverges by the nth term test. (Since $0 < \pi - e$, $\lim_{k\to\infty} k^{\pi-e} = \infty$.)

26. $\sum_{m=2}^{\infty} \frac{1}{(\ln 3)^m} = \frac{1}{(\ln 3)^2} \frac{1}{1-(1/\ln 3)} = \frac{1}{(\ln 3)(\ln 3 - 1)}.$

27.
$$\sum_{j=0}^{\infty} \left(\frac{1}{2^j} + \frac{1}{3^j}\right)^2 = \sum_{j=0}^{\infty} \left(\frac{1}{2^{2j}} + \frac{2}{6^j} + \frac{1}{3^{2j}}\right) = \sum_{j=0}^{\infty} \left(\frac{1}{4^j} + \frac{2}{6^j} + \frac{1}{9^j}\right) = \frac{4}{3} + \frac{12}{5} + \frac{9}{8} = \frac{583}{120}.$$

28. $\sum_{k=1}^{\infty} \left(\int_k^{k+1} \frac{dx}{x^2}\right) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$

29. Diverges by the nth term test: $\lim_{m\to\infty} \int_0^m e^{-x^2}\, dx = \sqrt{\pi}/2 \neq 0.$

30. Diverges by the nth term test: $\lim_{n\to\infty} \frac{\ln n}{\ln(3+n^2)} = \frac{1}{2} \neq 0.$

31. **No.** Using the series representation of $\sin x$ and the alternating series test, $\sin(1/n) > 1/n - 1/6n^3 = (6n^2-1)/6n^3 \geq 5/6n$ for all $n \geq 1$. Thus, $\sum_{n=1}^{\infty} \sin(1/n) > \frac{5}{6} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$

32. **Yes.** Since $0 < \sin(1/n) < 1/n$ for all $n \geq 1$, $0 < \sum_{n=1}^{\infty} \frac{1}{n} \sin(1/n) < \sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus, the given series converges by the comparison test.

33. **No.** Since $\lim_{n\to\infty} e^{-1/n} = 1 \neq 0$, the series diverges by the n-th term test.

34. **No.** Using the power series representation of e^x and the alternating series theorem,

$$1 - e^{-1/n} > \frac{1}{n} - \frac{1}{2n^2} = \frac{2n-1}{2n^2} \geq \frac{2n-n}{2n^2} = \frac{1}{2n}$$

for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right) \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}.$

35. (a) $\ln(n!) = \ln n + \ln(n-1) + \ln(n-2) + \cdots + \ln 2 + \ln 1$. Thus, (since $\ln 1 = 0$) $\ln(n!)$ is a right sum approximation so $\int_1^n \ln x\,dx$. Since $\ln x$ is an increasing function on the interval $[1, n]$, $\ln(n!) > \int_1^n \ln x\,dx = n\ln n - n + 1$. Therefore, $n! > e^{n\ln n - n+1} = n^n e^{1-n}$.

(b) Part (a) implies that $\dfrac{b^N}{N!} < \left(\dfrac{be}{N}\right)^N \dfrac{1}{e}$. Thus, $\dfrac{b^N}{N!} < \dfrac{1}{2}$ when $\left(\dfrac{be}{N}\right)^N < \dfrac{e}{2}$. Therefore, since $\dfrac{e}{2} > 1$, any $N > be$ satisfies the given inequality.

36. (a) For all $k \geq 1$, $0 < a_k < 1/k \implies \lim_{k\to\infty} a_k = 0$.

(b) **No**, the series diverges. Since $a_k = \displaystyle\int_k^\infty \frac{dx}{2x^2-1} \geq \int_k^\infty \frac{dx}{2x^2} = \frac{1}{2k}$,

$$\sum_{k=1}^{\infty} a_k \geq \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

(c) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges by the alternating series test — the terms of the series alternate in sign and are decreasing in magnitude (i.e., $a_{k+1} < a_k$).

37. $\left|\dfrac{a_{m+1}}{a_m}\right| = \dfrac{m^2+1}{(m+1)^2+1} \cdot |x| < 1 \implies R = 1$. The interval of convergence is $[-1, 1]$.

38. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left(\dfrac{n+1}{n}\right)^n \cdot (n+1) \cdot |x| < 1 \implies R = 0$. The interval of convergence is just $x = 0$.

39. The interval of convergence is $(-1/3, 1/3)$.

40. The interval of convergence is $(-\infty, \infty)$.

41. The interval of convergence is $[-1/3, 1/3)$.

42. The interval of convergence is $[-1/3, 1/3]$.

43. $(1, 5)$.

44. $(-\infty, \infty)$.

45. $[-3, 5)$.

46. $[0, 2)$.

47. $[-6, -4]$.

48. $[1/2, 3/2)$.

49. **Cannot** be true. The interval of convergence of a power series is symmetric around and includes its base point ($b = 1$ in this case).