Putnam Problem-Solving Seminar
Tuesday, November 27, 2007
Arithmetic Mean-Geometric Mean Inequality

Let $x_i > 0$ for $i = 1, 2, \ldots, n$. The *arithmetic mean* of $x_1, x_2, \ldots, x_n$ is

$$\frac{x_1 + x_2 + \ldots + x_n}{n}.$$

The *geometric mean* of $x_1, x_2, \ldots, x_n$ is

$$(x_1 x_2 \cdots x_n)^{1/n}.$$

The **arithmetic mean-geometric mean inequality** states that

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n},$$

with equality if and only if all of the x_i's are equal.

Hints. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra.

Problems.

1. If a, b, c are positive numbers such that
$$(1 + a)(1 + b)(1 + c) = 8,$$
prove that
$$abc \leq 1.$$

2. Prove the following inequality:
$$n\left[(n+1)^{1/n} - 1\right] < 1 + 1/2 + 1/3 + \cdots + 1/n < n - (n-1)n^{-1/(n-1)}.$$

3. Prove that the cube is the rectangular parallelepiped with maximum volume for a given surface area, and of minimum surface area for a given volume.

4. If a, b, c are positive numbers, prove that
$$(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2.$$

5. For positive numbers a and b, $a \neq b$, prove that
$$(ab^n)^{1/(n+1)} < \frac{a + nb}{n + 1}.$$

6. For each integer $n > 2$, prove that
$$n! < \left(\frac{n+1}{2}\right)^n.$$
7. For each integer $n > 2$, prove that $1 \times 3 \times 5 \cdots \times (2n-1) < n^n$.
8. Given that all roots of $x^6 - 6x^4 + ax^4 + bx^3 + cx^2 + dx + 1 = 0$ are positive, find a, b, c, d.

Some General Inequalities Problems.

1. Show that for positive numbers a, b, c,
$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$
2. Prove that for $0 < x < \pi/2$,
$$\cos^2 x + x \sin x < 2.$$
3. For each positive integer n, prove that
$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$
4. Prove that for any positive integer n,
$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n.$$
5. Prove that for any positive integer n,
$$\sqrt[n]{n} < 1 + \sqrt{2/n}.$$
6. Find all positive integers n such that
$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$