Arc Length

The general problem. Consider a function $y = f(x)$, and suppose that we want to find the total length of the graph of $f(x)$ from $x = a$ to $x = b$. For ease of notation, let's let L denote the length of the graph of $f(x)$ from $x = a$ to $x = b$. We'll assume that the curve is smooth in the sense that $f(x)$ and $f'(x)$ are both continuous. This guarantees that the curve has no sudden jumps or changes in direction.

The main idea. Approximate the curve with line segments, and approximate the length L of the curve by adding up the lengths of the approximating line segments. To find the *exact* value of the length of the curve, take the limit as the number of approximating line segments goes to infinity.

1. Partition the interval $[a, b]$ into n sub-intervals with endpoints

$$a = x_0,\ x_1,\ x_2,\ \ldots,\ x_n = b.$$

Let's construct the sub-intervals so that they all have equal width Δx.

2. Draw the line segments between the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ for all $i = 1, \ldots, n$. Let d_i denote the length of each such line segment. Then we *define* the arc length L as the following limit:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} d_i.$$

Question to think about: why does this definition make sense geometrically?

3. Use the distance formula to show that the length of each line segment is

$$\begin{aligned} d_i &= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\ \Delta x \end{aligned}$$

4. Note that the expression

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

looks almost like the definition of the derivative. In fact, the Mean Value Theorem for derivatives guarantees that there is a number $x_i^\star$ between x_{i-1} and x_i such that

$$f'(x_i^\star) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Thus

$$d_i = \sqrt{1 + (f'(x_i^\star))^2}\ \Delta x.$$

5. Thus, we obtain the following for L:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=1}^{n} d_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^\star))^2}\ \Delta x
\end{aligned}
$$

6. Finally, note that the previous expression is a Riemann sum, so the arc length L is given by:

$$
\begin{aligned}
L &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx \\
&= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx
\end{aligned}
$$

Summary. The length of the graph of the continuously differentiable function $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Examples.

1. Find the length of the curve

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}$$

from $x = 1$ to $x = 3$.

2. Find the length of the curve

$$f(x) = \sqrt{1 - x^2}$$

from $x = 0$ to $x = 1$. Use this result to find the circumference of a circle with radius 1.

3. Suppose that g is a function with the property that $g'(t) \geq 1$ if $a \leq t \leq b$. Let

$$f(x) = \int_a^x \sqrt{(g'(t))^2 - 1}\, dt.$$

Show that the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is $g(b) - g(a)$.