Math 112
Approximating Integrals in Maple

Recall that the definite integral $\int_a^b f(x)\,dx$ is defined to be the signed area of the region bounded by $x = a$, $x = b$, $y = f(x)$ and the x-axis. In many cases, we can evaluate the exact value of the definite integral using the Fundamental Theorem of Calculus. However, there are many definite integrals whose exact values are hard or impossible to find. Thus, a number of *numerical methods* have been developed to approximate definite integrals numerically.

We can numerically *approximate* the value of the definite integral $\int_a^b f(x)\,dx$ by using a *finite* number of approximating rectangles or trapezoids to approximate the area of the region in question. We've seen several examples in class of how to do compute these approximating sums by hand for small values of n (where n denotes the number of rectangles or trapezoids used in the approximation). Today, we'll work on evaluating approximating sums in Maple, and we'll think about how close the approximating sums get to the actual value of the integral. Using technology, we can use large values of n. Remember the notation that we introduced in class:

- L_n denotes the left rectangular sum approximation using n rectangles.
- R_n denotes the right rectangular sum approximation using n rectangles.
- M_n denotes the midpoint rectangular sum approximation using n rectangles.
- T_n denotes the trapezoid approximation using n trapezoids.

To get started with today's examples, pull up the Maple file posted on the Course Schedule page for Friday, February 1. The name of the file is *IntegralApprox*.

The command that you'll use for approximating definite integrals in Maple is *ApproximateInt*. The syntax for the *ApproximateInt* command is

ApproximateInt(f(x),x=a..b,partition=n,method=(left or right or midpoint or trapezoid),output=plot);

These inputs specify the following:

- $f(x)$ tells Maple the function whose integral you want to approximate.
- $x = a..b$ tells Maple the interval over which you want to approximate the integral.
- partition=n tells Maple how many sub-intervals you want to use for the approximation.
- method=(left or right or midpoint or trapezoid) tells Maple which approximating sum technique you want to use.

- output=plot tells Maple that you want to see a plot of the approximating rectangles or trapezoids (as well as the approximate value of the integral).

For example, to approximate $\int_1^5 x^2\,dx$ using a left rectangular sum approximation with $n = 8$ sub-intervals, you would use the following command:

ApproximateInt($x^2, x = 1..5$, partition=8,method=left,output=plot);

1. Consider the integral $I = \int_1^5 x^2\,dx$.
   (a) Is the function $f(x) = x^2$ increasing, decreasing, or neither on the interval $[1, 5]$?
   Solution. $f(x) = x^2$ is increasing on the interval $[1, 5]$.
   (b) Find the values of the approximating sums L_8 and R_8.
   Solution. $L_8 = 35.5$ and $R_8 = 47.5$. See the Maple file posted on the Course Schedule page.
   (c) Without finding the actual value of the integral, write an inequality statement that compares I, L_8, and R_8.
   Solution. Looking at the plots in Maple, we see that L_8 is an under-approximation and R_8 is an over-approximation. Thus
   $$L_8 \leq I \leq R_8.$$
   So we know that the actual value of the integral is somewhere between 35.5 and 47.5. We have *trapped* the exact value of the integral between 35.5 and 47.5, i.e.
   $$35.5 \leq I \leq 47.5.$$
   If we increase the number of approximating rectangles used in the left and right rectangular sum approximations, we could trap I more closely.
   (d) Find the actual value of the integral using the Fundamental Theorem of Calculus. Remember that you can check your answer in Maple. Confirm that your inequality statement from the previous question is valid.
   Solution.
   $$I = \int_1^5 x^2\,dx = \frac{1}{3}x^3\bigg|_1^5 = \frac{124}{3} = 41\frac{1}{3},$$
   so the inequality $L_8 \leq I \leq R_8$ is valid.

2. Consider the integral $I = \int_0^2 e^{x^2}\,dx$. Note that this integral can't be evaluated by anti-differentiation, so we'll have to use numerical methods to approximate the value of the integral. To enter e^{x^2} in Maple, use $exp(x^2)$.
   (a) Is $f(x) = e^{x^2}$ increasing, decreasing, or neither on the interval $[0, 2]$?
   (b) Compute L_{25} and R_{25}.
   (c) Write an inequality statement that compares I, L_{25}, and R_{25}.
   (d) Compute L_8 and R_{15}.
   (e) Write an inequality statement that compares I, L_8, and R_{15}.

3. Consider the integral $I = \int_1^4 \ln x \, dx$.
   (a) Is $f(x) = \ln x$ increasing, decreasing, or neither on the interval $[1, 4]$?
   (b) Compute L_{14} and R_{14}.
   (c) Write an inequality statement that compares I, L_{14}, and R_{14}.
   (d) Compute L_{12} and R_5.
   (e) Write an inequality statement that compares I, L_{12}, and R_5.

4. **Trapping the integral of an increasing function.** Suppose that $f(x)$ is an *increasing* function on the interval $[a, b]$. Let

$$I = \int_a^b f(x)\, dx,$$

   i.e. I denotes the actual value of the integral $\int_a^b f(x)\, dx$. Let n and m be any positive integers (i.e. n and m represent any number of approximating rectangles). Make a conjecture about an inequality statement that compares I, L_n, and R_m. Explain your conjecture. Hint: Draw a picture. If necessary, do more examples.

5. Consider the integral $I = \int_1^4 \frac{1}{x} \, dx$.
   (a) Is $f(x) = \frac{1}{x}$ increasing, decreasing, or neither on the interval $[1, 4]$?
   (b) Compute L_{22} and R_{22}.
   (c) Write an inequality statement that compares I, L_{22}, and R_{22}.
   (d) Compute L_7 and R_{15}.
   (e) Write an inequality statement that compares I, L_7, and R_{15}.

6. Consider the integral $I = \int_0^2 e^{-x^2} \, dx$.
   (a) Is $f(x) = e^{-x^2}$ increasing, decreasing, or neither on the interval $[0, 2]$?
   (b) Compute L_{30} and R_{30}.
   (c) Write an inequality statement that compares I, L_{30}, and R_{30}.
   (d) Compute L_{18} and R_{10}.
   (e) Write an inequality statement that compares I, L_{18}, and R_{10}.

7. **Trapping the integral of a decreasing function.** Suppose that $f(x)$ is a *decreasing* function on the interval $[a, b]$. Let

$$I = \int_a^b f(x)\, dx,$$

   i.e. I denotes the actual value of the integral $\int_a^b f(x)\, dx$. Let n and m be any positive integers (i.e. n and m represent any number of approximating rectangles). Make a conjecture about an inequality statement that compares I, L_n, and R_m. Explain your conjecture. Hint: Draw a picture. If necessary, do more examples.

8. **Using the trapping idea to estimate an integral.** Consider the integral $I = \int_0^1 \sin(x^2)\, dx$. Note that the exact value of I cannot be computed by anti-differentiation. Use the idea of trapping the integral to trap the value of I to within 0.01. This means that we want to find a number a such that the inequality statement $a \le I \le a + 0.01$ is valid.

9. **The divide and conquer technique.** Consider the integral

$$I = \int_0^2 \sin(x^2)\, dx.$$

The function $f(x) = \sin(x^2)$ is neither strictly decreasing nor strictly increasing on the interval $[0, 2]$. Can we still use left and right approximating sums to trap the exact value of the integral?

10. Consider the integral $I = \int_1^4 x^2\, dx$.

    (a) Compute L_5, R_5, and T_5. What relationship do you observe among these three numbers?

    (b) Compute L_{10}, R_{10}, and T_{10}. What relationship do you observe among these three numbers?

11. Make a conjecture about the relationship between L_n, R_n, and T_n.

12. Consider the integral $I = \int_0^1 (1 - x^4)\, dx$.

    (a) Compute L_{10}, R_{10}, T_{10}, and M_{10}.

    (b) Compute the exact value of the integral.

    (c) Which approximation method is the most accurate in this example? Which method gives the second best approximation?