# Absolute Convergence and Alternating Series

**Definition: Absolute and conditional convergence.** Let $\Sigma a_k$ be any series. (The terms $a_k$ may be negative or positive.)

- If $\Sigma|a_k|$ converges, then we say that $\Sigma a_k$ **converges absolutely**.
- If $\Sigma|a_k|$ diverges but $\Sigma a_k$ converges, then we say that $\Sigma a_k$ **converges conditionally**.

**Theorem.** If $\Sigma|a_k|$ converges, so does $\Sigma a_k$.

**Example 1.** Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges conditionally, converges absolutely, or diverges.

**Example 2.** Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3+2k+1}$ converges conditionally, converges absolutely, or diverges.

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**Definition.** An **alternating series** is a series whose terms alternate in sign, i.e. a series of the form

$$\sum_{k=1}^{\infty}(-1)^{k+1}c_k = c_1 - c_2 + c_3 - c_4 + \cdots ,$$

where each $c_k$ is positive.

**Alternating Series Test.** Consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}c_k = c_1 - c_2 + c_3 - c_4 + \cdots .$$

Suppose that the following two conditions are satisfied:

- $c_1 \geq c_2 \geq c_3 \geq \cdots 0$, i.e. $c_{n+1} \leq c_n \geq 0$ for all $n$
- $\displaystyle\lim_{k\to\infty} c_k = 0$

Then $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}c_k$ converges.

**Example 3.** Determine whether the alternating harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally, converges absolutely, or diverges.

**Example 4.** Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$ converges conditionally, converges absolutely, or diverges.

**Example 5.** Determine whether the series $\sum_{k=5}^{\infty} \frac{(-1)^k \ln k}{k}$ converges conditionally, converges absolutely, or diverges.