§8.4 Miscellaneous Antiderivatives

1. $\int \frac{\sin x}{(3+\cos x)^2}\,dx = \frac{1}{3+\cos x} + C.$
[substitution: $u = 3 + \cos x$.]

2. $\int \frac{x^2}{x+1}\,dx = \frac{1}{2}(x+1)^2 - 2(x+1) + \ln|x+1| + C.$
[substitution: $u = x + 1$.]

3. $\int x\left(3+4x^2\right)^5\,dx = \frac{1}{48}\left(3+4x^2\right)^6 + C.$
[substitution: $u = 3 + 4x^2$.]

4. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$

5. $\int \frac{x}{\sqrt[3]{x^2+4}}\,dx = \frac{3}{4}\left(x^2+4\right)^{2/3} + C.$
[substitution: $u = x^2 + 4$.]

6. $\int \frac{dx}{x(3x-2)} = \int \left(\frac{3}{2(3x-2)} - \frac{1}{2x}\right) dx = \left(\ln|3x-2| - \ln|x|\right)/2 + C.$

7. $\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}\left(\ln|x|\right)^3 + C.$
[substitution: $u = \ln x$.]

8. $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$
[integration by parts (twice).]

9. $\int \frac{\ln x}{x}\,dx = \frac{1}{2}\left(\ln|x|\right)^2 + C.$
[substitution: $u = \ln x$] .

10. $\int x\sqrt{x+2}\,dx = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$
[substitution: $u = x + 2$.]

11. $\int \frac{x}{3x+2}\,dx = \frac{1}{3}\int \left(1 - \frac{2}{3x+2}\right) dx = \frac{x}{3} - \frac{2}{9}\ln|3x+2| + C.$

12. Using integration by parts (with $u = x$ and $dv = \cos x\,dx$)
$\int x\cos x\,dx = x\sin x + \cos x + C.$

13. $\int \sin^2(3x)\cos(3x)\,dx = \frac{1}{9}\sin^3(3x) + C.$
[substitution: $u = \sin(3x)$.]

14. $\int xe^{3x}\,dx = \frac{1}{3}e^{3x}\left(x - \frac{1}{3}\right) + C.$
[integration by parts: $u = x$, $dv = e^{3x}\,dx$.]

15. $\int xe^{3x^2}\,dx = \frac{1}{6}e^{3x^2} + C.$
[substitution: $u = 3x^2$.]

16. $\int \frac{dx}{1+4x^2} = \frac{1}{2}\arctan(2x) + C.$
[substitution: $u = 2x$.]

17. $\int (2-3x)^{10}\,dx = -\frac{1}{33}(2-3x)^{11} + C.$
[substitution: $u = 2 - 3x$.]

18. $\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$
[integration by parts: $u = \arctan x$, $dv = dx$.]

19. $\int \frac{\sec^2 x}{3+\tan x}\,dx = \ln|3+\tan x| + C.$
[substitution: $u = 3 + \tan x$.]

20. $\int x\sin x\,dx = \sin x - x\cos x + C.$
[integration by parts: $u = x$, $dv = \sin x\,dx$.]

21. $\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$
[partial fractions: $\frac{1}{(x-1)(x+2)} = \frac{1}{3(x-1)} - \frac{1}{3(x+2)}$.]

22. $\int x^2\ln x\,dx = \frac{1}{3}x^3\ln|x| - \frac{1}{9}x^3 + C.$
[integration by parts: $u = \ln x$, $dv = x^2\,dx$.]

23. $\int \frac{2x+3}{4x+5}\,dx = \frac{1}{2}\int\left(1 + \frac{1}{4x+5}\right)dx = \frac{x}{2} + \frac{\ln|4x+5|}{8} + C.$

24. $\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + \arctan x + C.$

25. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \arcsin\left(e^x\right) + C.$
[substitution: $u = e^x$.]

26. $\int \frac{\sin x}{2+\cos x}\,dx = -\ln(2+\cos x) + C.$
[substitution: $u = 2 + \cos x$.]

27. $\int \ln x\,dx = x(\ln x - 1) + C.$
[integration by parts: $u = \ln x$, $dv = dx$.]

28. $\int x\cos\left(3x^2\right)\,dx = \frac{1}{6}\sin(3x^2) + C.$
[substitution: $u = 3x^2$.]

29. $\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C.$
[integration by parts: $u = \arcsin x$, $dv = dx$.]

30. Let $u = \sqrt{x}$. Then,

$$\int \frac{\sqrt{x}}{1+x}\,dx = 2\int \frac{u^2}{1+u^2}\,du = 2\int\left(1 - \frac{1}{1+u^2}\right)du$$
$$= 2u - 2\arctan u = 2\sqrt{x} - 2\arctan(\sqrt{x}) + C.$$

31. $\int \frac{dx}{x^2+2x+3} = \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C.$
[complete the square, substitution: $x^2 + 2x + 3 = (x+1)^2 + 2 = u^2 + 2$.]

32. $\int \frac{x}{\sqrt{x-2}}\,dx = \frac{2}{3}(x-2)^{3/2} + 4\sqrt{x-2} + C.$
[substitution: $u = x - 2$.]

33. $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\arcsin(2x) + C.$
[substitution: $u = 2x$.]

34. $\int \frac{x^3}{1+x^2}\,dx = \frac{1}{2}(1+x^2) - \frac{1}{2}\ln(1+x^2) + C.$
[substitution: $u = 1 + x^2$.]

35. $\int \tan x\,dx = -\ln|\cos x| + C.$
[Write $\tan x = \sin x/\cos x$, then use the substitution $u = \cos x$.]

36. $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$

37. $\int e^{2x}\sqrt{1+e^x}\,dx = \frac{2}{5}\left(1+e^x\right)^{5/2} - \frac{2}{3}\left(1+e^x\right)^{3/2} + C.$
[substitution: $u = 1 + e^x$.]

38. $\int \frac{dx}{1+x^2} = \arctan x + C.$

39. Let $u = (x+1)/2$. Then, $du = \frac{1}{2}dx$ and
$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u = \arcsin\left(\frac{1}{2}(x+1)\right) + C.$

40. Using integration by parts with $u = \arcsin x$ and $dv = x^2\,dx$,

$$\int x^2 \arcsin x\,dx = \frac{1}{3}x^3 \arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

To find the remaining antiderivative, use the substitution $w = 1 - x^2$:

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1-w}{\sqrt{w}}\,dw = -\frac{1}{2}\int \frac{dw}{\sqrt{w}} + \frac{1}{2}\int \sqrt{w}\,dw$$
$$= -\sqrt{w} + \frac{1}{3}w^{3/2} + C = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.$$

Therefore, $\int x^2 \arcsin x\,dx = \frac{1}{3}x^3 \arcsin x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C.$

41. Let $u = \ln x$. Then, $du = dx/x$ and $\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln|x|} + C.$

42. $\int x \arctan x\,dx = \frac{1}{2}(x^2+1)\arctan x - \frac{x}{2} + C.$
[integration by parts: $u = \arctan x$, $dv = x\,dx$.]

43. $\int \frac{dx}{9x^2-4} = \int \frac{dx}{(3x-2)(3x+2)} = \frac{1}{4}\int \left(\frac{1}{3x-2} - \frac{1}{3x+2}\right)dx$
$= \left(\ln|3x-2| - \ln|3x+2|\right)/12 + C.$

44. $\int \frac{x+5}{x^2+3x-4}\,dx = \frac{6}{5}\ln|x-1| - \frac{1}{5}\ln|x+4| + C.$
[partial fractions: $\frac{x+5}{x^2+3x-4} = \frac{6}{5(x-1)} - \frac{1}{5(x+4)}$.]

45. $\int \frac{x^3}{\sqrt{4-x^2}}\,dx = -4\sqrt{4-x^2} + \frac{1}{3}\left(4-x^2\right)^{3/2} + C.$
[substitution ($u = 4 - x^2$) or integration by parts ($u = x^2$, $dv = x/\sqrt{4-x^2}\,dx$).]

46. $\int \frac{dx}{\sqrt[3]{x-1}} = \frac{3}{2}(x-1)^{2/3} + C.$

47. $\int \frac{x}{(x-1)(x+1)}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| = \frac{1}{2}\ln\left|x^2-1\right| + C.$
[partial fractions: $\frac{x}{(x-1)(x+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)}$.]

48. $\int x^3 e^{x^2}\,dx = \frac{1}{2}e^{x^2}(x^2-1) + C.$
[substitution ($w = x^2$), then integration by parts ($u = w, dv = e^w\,dw$).]

49. $\int \frac{dx}{\sqrt{9+x^2}} = \ln\left|x + \sqrt{9+x^2}\right| + C.$
[trigonometric substitution: $x = 3\tan t$.]

50. $\int \frac{dx}{2x-x^2} = \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x-2| = \frac{1}{2}\ln\left|\frac{x}{x-2}\right| + C.$
[partial fractions: $\frac{1}{2x-x^2} = \frac{1}{x(2-x)} = \frac{1}{2x} + \frac{1}{2(2-x)}$.]

51. $$\int \frac{x^2}{1-3x}\,dx = -\frac{1}{54}(1-3x)^2 + \frac{2}{27}(1-3x) - \frac{1}{27}\ln|1-3x| + C$$
$$= -\frac{1}{6}x^2 - \frac{1}{9}x - \frac{1}{27}\ln|1-3x| + C.$$
[substitution ($u = 1-3x$) or partial fractions.]

52. $\int \frac{x}{\left(x^2-1\right)^3}\,dx = -\frac{1}{4\left(x^2-1\right)^2} + C.$
[substitution: $u = x^2 - 1$.]

53. $\int e^x e^{2x}\,dx = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C.$

54. $\int \sqrt{4x-3}\,dx = \frac{1}{6}(4x-3)^{3/2} + C.$
[substitution: $u = 4x - 3$.]

55. $\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2x + 2\arctan x + C.$
[integration by parts: $u = \ln(1+x^2), dv = dx$.]

56. $\int \sin(\sqrt{x})\,dx = 2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C.$
[substitution ($w = \sqrt{x}$), then integration by parts ($u = w, dv = \sin w\,dw$.]

57. Using integration by parts $\int x \arcsin x\, dx = \dfrac{x^2 \arcsin x}{2} - \dfrac{1}{2}\int \dfrac{x^2}{\sqrt{1-x^2}}\, dx$. Now, using the trigonometric substitution $x = \sin t$,

$$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \int \sin^2 t\, dt = \left(-\sin t \cos t + t\right)/2$$
$$= \left(-x\sqrt{1-x^2} + \arcsin x\right)/2.$$

Thus,

$$\int x \arcsin x\, dx = \frac{x^2 \arcsin x}{2} + \frac{x\sqrt{1-x^2}}{4} - \frac{\arcsin x}{4} + C.$$

58. $\displaystyle\int \frac{dx}{9-x^2} = \frac{1}{6}\int\left(\frac{1}{3+x} + \frac{1}{3-x}\right) dx = \left(\ln|x+3| - \ln|3-x|\right)/6.$

59. Let $u = 2x+3$. Then, $\displaystyle\int \frac{dx}{\sqrt{2x+3}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{2x+3} + C.$

60. $\displaystyle\int \frac{dx}{\left(4-x^2\right)^{3/2}} = \frac{x}{4\sqrt{4-x^2}} + C.$
[trigonometric substitution: $x = 2\sin t$.]

61. Let $u = 2x+3$. Then,

$$\int \frac{x}{(2x+3)^4}\, dx = \frac{1}{4}\int \frac{u-3}{u^4}\, du = -\frac{1}{8u^2} + \frac{1}{4u^3}$$
$$= -\frac{1}{8(2x+3)^2} + \frac{1}{4(2x+3)^3} = -\frac{2x+1}{8(2x+3)^3} + C.$$

62. $\displaystyle\int x\sqrt{2x+1}\, dx = \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$
[substitution: $u = 2x+1$.]

63. $\displaystyle\int \frac{\tan x}{\sec^2 x}\, dx = -\frac{1}{2}\cos^2 x + C.$
[Write $\frac{\tan x}{\sec^2 x} = \sin x \cos x$, then use substitution ($u = \cos x$).]

64. $\displaystyle\int \frac{x}{16+9x^2}\, dx = \frac{1}{18}\ln\left(16+9x^2\right) + C.$
[subsitution: $u = 16+9x^2$.]

65. $\displaystyle\int \frac{dx}{e^x - 1} = \ln\left|1-e^{-x}\right| + C.$
[Write $\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}}$, then use subsitution $u = 1-e^{-x}$.]

66. $\int \frac{dx}{\sqrt{2x-x^2}} = -\arcsin(1-x) + C.$
[Write $2x - x^2 = 1 - (1-x)^2$, then substitute $u = 1 - x$.]

67. $\int \frac{dx}{1+\sqrt{x}} = 2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C.$
[subsitution: $u = 1 + \sqrt{x}$.]

68. $\int \frac{x^3\,dx}{(x^2+1)^2} = \frac{1}{2(x^2+1)} + \frac{1}{2}\ln(x^2+1) + C.$
[substitution $u = x^2 + 1$.]

69. $\int x^2 \ln(3x)\,dx = \frac{1}{3}x^3\ln(3x) - \frac{1}{9}x^3 + C.$
[integration by parts: $u = \ln(3x)$, $dv = x^2\,dx$.]

70. $\int \frac{x}{9+4x^4}\,dx = \frac{1}{12}\arctan\left(\frac{2x^2}{3}\right) + C.$

71. $\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln|x| - \frac{4}{9}x^{3/2} + C.$
[integration by parts: $u = \ln x$, $dv = \sqrt{x}\,dx$.]

72. $\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C.$
[integration by parts: $u = x$, $dv = \sec^2 x\,dx$.]

73. $\int \frac{7-x}{(x+3)(x^2+1)}\,dx = \ln|x+3| + 2\arctan x - \frac{1}{2}\ln(x^2+1) + C.$
[partial fractions: $\frac{7-x}{(x+3)(x^2+1)} = \frac{1}{x+3} + \frac{2-x}{x^2+1}$.]

74. $\int \frac{x+6}{(x+1)(x^2+4)}\,dx = \ln|x+1| + \arctan(x/2) - \frac{1}{2}\ln(x^2+4) + C.$
[partial fractions: $\frac{x+6}{(x+1)(x^2+4)} = \frac{1}{x+1} + \frac{2-x}{x^2+4}$.]

75. $$\begin{aligned}\int x\sin^2 x\cos x\,dx &= \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x \\ &= \frac{1}{3}x\sin^3 x + \frac{1}{9}\sin^2 x\cos x + \frac{2}{9}\cos x + C.\end{aligned}$$
[integration by parts with $u = x$ and $dv = \sin^2 x\cos x\,dx$.]

76. Let $u = \sin x$. Then, $du = \cos x\,dx$ and

$$\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cos x\,dx = \int \sin^3 x(1 - \sin^2 x)\cos x\,dx$$
$$= \int \sin^3 x \cos x\,dx - \int \sin^5 x \cos x\,dx = \int u^3\,du - \int u^5\,du$$
$$= \tfrac{1}{4}u^4 - \tfrac{1}{6}u^6 + C = \tfrac{1}{4}\sin^4 x - \tfrac{1}{6}\sin^6 x + C.$$

77. $\displaystyle\int \frac{dx}{x^3 + x} = \ln|x| - \frac{1}{2}\ln(x^2 + 1) + C.$
[partial fractions: $\frac{1}{x^3+x} = \frac{1}{x} - \frac{x}{x^2+1}$.]

78. $\displaystyle\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$

79. $\displaystyle\int \left(x^2 + 2x + 3\right)^{3/2} dx = \frac{1}{4}(x+1)(x^2 + 2x + 3)^{3/2} + \frac{3}{4}(x+1)\sqrt{x^2 + 2x + 3} +$
$\displaystyle\frac{3}{2}\ln\left|\frac{\sqrt{x^2 + 2x + 3}}{\sqrt{2}} + \frac{x+1}{\sqrt{2}}\right| + C.$
[Write $x^2 + 2x + 3 = (x+1)^2 + 2$, then use a trigonometric substitution ($x + 1 = \sqrt{2}\tan t$).]

80. $\displaystyle\int \sin(3x)\cos(5x)\,dx = \frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.$
[Write $\sin(3x)\cos(5x) = \frac{1}{2}\sin(8x) - \frac{1}{2}\sin(2x)$.]

81. Let $u = \sqrt{1 + e^x}$. Then,

$$\int \sqrt{1 + e^x}\,dx = 2\int \frac{u^2}{u^2 - 1}\,du = 2\int \left(1 + \frac{1}{u^2 - 1}\right)du$$
$$= 2\int \left(1 - \frac{1}{2(u+1)} + \frac{1}{2(u-1)}\right)du$$
$$= 2u - \ln|u + 1| + \ln|u - 1|$$
$$= 2\sqrt{1 + e^x} - \ln\left|\sqrt{1 + e^x} + 1\right| + \ln\left|\sqrt{1 + e^x} - 1\right| + C.$$

82. $\displaystyle\int \frac{dx}{x\left(x + \sqrt[3]{x}\right)} = -\frac{3}{\sqrt[3]{x}} - 3\arctan(\sqrt[3]{x}) + C.$
[substitution ($u = x^{1/3}$), then partial fractions ($\frac{1}{u^4+u^2} = \frac{1}{u^2} - \frac{1}{u^2+1}$).]

83. $\displaystyle\int \frac{dx}{x^3 + 1} = \frac{1}{3}\ln|x + 1| - \frac{1}{6}\ln\left|x^2 - x + 1\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x - 1}{\sqrt{3}}\right) + C.$
[Write $x^3 + 1 = (x+1)(x^2 - x + 1) = (x+1)\cdot\left((2x-1)^2 + 3\right)/4$, then use partial fractions, etc.]

84. $\displaystyle\int \frac{dx}{(e^x - e^{-x})^2} = \frac{1}{4}\left(\frac{1}{1+e^x} + \frac{1}{1-e^x}\right) = \frac{1}{2-2e^{2x}} + C.$

[Write $\left(e^x - e^{-x}\right)^{-2} = e^{2x}\left(e^{2x} - 1\right)^{-2} = (e^x)^2\,(e^x - 1)^{-2}\,(e^x + 1)^{-2}$, then use substitution ($u = e^x$) and partial fractions.]

85. $\displaystyle\int x \tan^2 x\, dx = x \tan x - \frac{1}{2}x^2 - \ln|\sec x| + C.$

[integration by parts: $u = x$, $dv = \tan^2 x\, dx$.]

86. $\displaystyle\int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C.$

[Write $\cos^3 x = \cos x \cos^2 x = \cos x - \cos x \sin^2 x$, then use the substitution $u = \sin x$.]

87. $\displaystyle\int \sin x \sin(2x)\, dx = \frac{1}{2}\sin x - \frac{1}{6}\sin(3x) + C.$

[Write $\sin x \sin(2x) = \frac{1}{2}\cos x - \frac{1}{2}\cos(3x)$.]

88. $\displaystyle\int \sin^5 x \cos^2 x\, dx = \frac{1}{7}\sin^6 x \cos x - \frac{1}{35}\sin^4 x \cos x - \frac{4}{105}\sin^2 x \cos x - \frac{8}{105}\cos x + C.$

[Write $\cos^2 x = 1 - \sin^2 x$, then use a reduction formula.]