Math 347
Daily Objectives
Class Session 1
Tuesday, August 28, 2007

1. Introduction to mathematical modeling

   Example: Lifeboats and life vests.

   In a January, 1998 article entitled *Most Ferries Short of Life Rafts–Is it a Titanic Problem or a Reasonable Cost-Control Measure?*, the Seattle Times announced that Washington State Ferries, which service the Puget Sound area, were not carrying enough lifeboats for all passengers. In fact, there is a reason for this decision. The ferries do carry enough life vests to make up the difference. However, the Washington State ferries traverse very cold waters, so it would be better to carry more lifeboats than life vests, since a passenger is much more likely to survive in a lifeboat than in cold water. However, lifeboats require more storage space than life vests, so there is a tradeoff. The capacity of the ferries would be greatly reduced if the ferries had to carry enough lifeboats for every passenger.

   Suppose the XYZ Boat Company has come to us with the following problem. They are designing a new ferry boat, and want to determine the number of lifeboats and life vests to equip it with. They give us the following information:

   - Each vest holds 1 person and requires 0.05 m^3 of storage space.
   - Each boat holds 20 people and requires 2.1 m^3 of storage space.
   - There must be capacity for 1000 passengers.
   - The total storage space is at most 85 m^3.

   How many of each should they install on the new ferry? One might go through the following steps to solve this problem.

   (a) First determine whether or not there is enough space to accommodate all passengers on lifeboats. This would require $1000/20 = 50$ lifeboats, and $50 \times 2.1 = 105$ m^3 of storage space, so we will have to use some combination of lifeboats and life vests.

   (b) Next, let's make sure that there is enough space for life vests for all 1000 passengers; otherwise, the problem is impossible. The total storage space required for 1000 life vests is $1000 \times 0.05 = 50$ m^3, so it would be possible to provide only life vests and have storage space left over. However, for an optimal solution, we'd like to use as many lifeboats as possible.

(c) To use as many lifeboats as possible, we must use all of the available storage space and provide exactly the necessary capacity. We can set this up as a mathematical problem. Let

x_1 = number of life vests
x_2 = number of lifeboats

There are two equations to be satisfied, the first stating that the total capacity must be 1000 and the second stating that the total storage volume must be 85 m^3:

$$\begin{aligned} x_1 + 20x_2 &= 1000 \\ 0.05x_1 + 2.1x_2 &= 85 \end{aligned} \tag{1}$$

(d) Since Eqn. (1) is just a linear system of equations, we can easily solve it (using, say, Maple) to obtain

x_1 = 363.6364
x_2 = 31.8182

(e) Next, we interpret these results in terms of the original problem. Clearly, we can't provide fractional vests or lifeboats, so we recommend that the XYZ Boat Company designs the new ferries with 31 lifeboats and $1000 - 31 \times 20 = 380$ life vests.

(f) Upon presenting this information to the XYZ Boat Company, we are immediately told that the lifeboats are always arranged on symmetric racks on the two sides of the ferry, so there must be an even number of lifeboats. Thus, we recommend using 30 lifeboats and 400 life vests.

(g) However, we notice that 31.8182 is almost equal to 32. This suggests that if we had just a little bit more storage space, we could increase the number of life boats to 32. Using 32 lifeboats and 360 life vests would require 85.2 m^3 of storage space, so we ask the XYZ Boat Company if they can find an additional 0.2 m^3 of storage space. They respond by telling us that the 85 m^3 figure was just an approximation, and they can easily find another 0.2 m^3 if it means they can fit in another 2 lifeboats.

2. The Five-Step Method

- Ask the question. Make a list of all variables and constants, including appropriate units. State any assumptions about the variables, including equations and inequalities.
- Select a modeling approach.
- Formulate the mathematical model.
- Solve the model using mathematical techniques.

- Answer the original question in non-technical terms, keeping in mind that the information presented is not always complete or accurate.

3. One-Variable Optimization and Sensitivity Analysis

   The *sensitivity* of x_c to a parameter α is given by

$$S(x_c, \alpha) = \frac{dx_c}{d\alpha} \cdot \frac{\alpha}{x_c}.$$