Math 333

Quiz 9 solutions.

1. $y'' - xy' - y = 0$.

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n \qquad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^r$

$y'' - xy' - y = 0$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - na_n - a_n\right] x^n = 0$$

$$\Rightarrow a_2 = \frac{1}{2} a_0, \quad a_{n+2} = \frac{(n+1)a_n}{(n+1)(n+2)} = \frac{a_n}{n+2}$$

Thus, the general recurrence relation is:

$$a_{n+2} = \frac{a_n}{n+2}.$$

Recall that a_0 and a_1 are arbitrary constants.

$$a_2 = \frac{a_0}{2} \qquad a_4 = \frac{a_2}{4} = \frac{a_0}{2\cdot 4} \qquad a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}$$

$$a_3 = \frac{a_1}{3} \qquad a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5} \qquad a_7 = \frac{a_5}{7} = \frac{a_1}{3\cdot 5\cdot 7}$$

$$\Rightarrow y(x) = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2\cdot 4}x^4 + \frac{a_1}{3\cdot 5}x^5$$

$$+ \frac{a_0}{2\cdot 4\cdot 6}x^6 + \frac{a_1}{3\cdot 5\cdot 7}x^7 + \cdots$$

In general, observe that:

$$a_{2n} = \frac{a_0}{2^n \cdot n!} \qquad a_{2n+1} = \frac{a_1 \cdot 2^n \cdot n!}{(2n+1)!}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{2^n n!} x^{2n} + \sum_{n=0}^{\infty} \frac{a_1 \cdot 2^n \cdot n!}{(2n+1)!} x^{2n+1}$$

2. $(1+x^2)y'' - 4xy' + 6y = 0.$

$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n \quad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$

$$(1+x^2)\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - 4x\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$+ 6\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^{n+2}$$

$$- \sum_{n=0}^{\infty} 4(n+1)a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 6a_n\right] x^n + \sum_{n=2}^{\infty} (n-1)\cdot n\cdot a_n x^n$$

$$- \sum_{n=1}^{\infty} 4\cdot n\cdot a_n x^n = 0.$$

$$(2a_2 + 6a_0) + (2\cdot 3\cdot a_3 + 6a_1)x + \sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2} + 6a_n\right]x^n$$

$$+ \sum_{n=2}^{\infty} (n-1)\cdot n\cdot a_n x^n - 4\cdot 1\, a_1 x - \sum_{n=2}^{\infty} 4n a_n x^n = 0.$$

$$\Rightarrow (2a_2 + 6a_0) + (6a_3 + 6a_1 - 4a_1)x$$
$$+ \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} + 6a_n + (n-1)\cdot n \cdot a_n - 4na_n]x^n = 0$$

$$\Rightarrow (2a_2 + 6a_0) + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} + (6 + (n-1)\cdot n - 4n)a_n]x^n = 0$$

$$\Rightarrow 2a_2 + 6a_0 = 0 \Rightarrow a_2 = -3a_0$$
$$2a_1 + 6a_3 = 0 \Rightarrow a_3 = -\frac{1}{3}a_1$$

$$(n+1)(n+2)a_{n+2} + (6 + (n-1)\cdot n - 4n)a_n = 0$$

$$a_{n+2} = \frac{(4n - n(n-1) - 6)a_n}{(n+1)(n+2)} = \frac{(n-2)(n-3)a_n}{(n+1)(n+2)}.$$

Recall that a_0 and a_1 are arbitrary constants.

$a_2 = -3a_0$, $a_3 = -\frac{1}{3}a_1$

$a_4 = 0 \Rightarrow a_6 = a_8 = \cdots = 0$ $a_{2n} = 0$ for $n \geq 2$

$a_5 = 0 \Rightarrow a_7 = a_9 = \cdots = 0$ $a_{2n+1} = 0$ for $n \geq 2$

The general solution of the DE is:

$$y(x) = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3$$