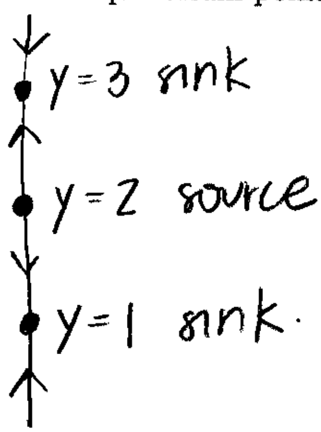
Math 333 Solutions.
Quiz 2
Thursday, January 31, 2008

1. Sketch the phase line for the differential equation

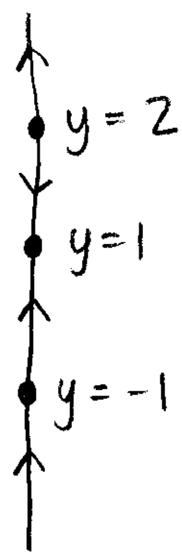
$$\frac{dy}{dt} = -(y-1)(y-2)(y-3).$$

Identify each equilibrium point as a sink, source, or node.



2. Find a polynomial function $f(y)$ of degree 4 such that the autonomous DE $\frac{dy}{dt} = f(y)$ has a node at $y=-1$, a sink at $y=1$, a source at $y=2$, and no other equilibrium points.

$$f(y) = (y+1)^2(y-1)(y-2)$$



3. Consider the differential equation

$$\frac{dy}{dt} = -y^2.$$

(a) Show that

$$y_1(t) = \frac{1}{t-1} \text{ and } y_2(t) = \frac{1}{t-2}$$

are both solutions of the given differential equation.

(b) Let $y(t)$ be any solution of the differential equation with initial condition $y(0) = y_0$ that satisfies $-1 < y_0 < -\frac{1}{2}$. Why are we guaranteed that such a solution $y(t)$ always exists?

(c) Without explicitly solving the differential equation, find a lower and upper bound (in terms of t) for $y(t)$. Explain how you obtained these bounds, and if you apply any theorems, be sure that you explicitly state why those theorems apply to this problem.

(d) Describe the behavior of $y(t)$ as $t \to \infty$ and as $t \to -\infty$.

(a) $\frac{dy_1}{dt} = -1(t-1)^{-2} = -\frac{1}{(t-1)^2} = -y_1^2$

$\frac{dy_2}{dt} = -1(t-2)^{-2} = -\frac{1}{(t-2)^2} = -y_2^2$

(b) $\frac{dy}{dt} = -y^2$ is continuous for all t, y, so the Existence Theorem applies.

(c) Since $\frac{dy}{dt} = -y^2$ and $\frac{\partial f}{\partial y} = -2y$ are both continuous for all t, y, the Uniqueness Theorem applies. Thus, since

$$-1 = y_1(0) < y(0) < y_2(0) = -\frac{1}{2},$$

we have

$$y_1(t) = \frac{1}{t-1} < y(t) < \frac{1}{t-2} \text{ for all } t.$$

(d) Note that $\frac{dy}{dt} < 0$ for all t, y, so $y(t)$ is always decreasing. As $t \to \infty$ and $t \to -\infty$, $y(t)$ approaches the equilibrium solution $y = 0$.