Math 333
Quiz 1
Tuesday, January 22, 2008

Solutions.

1. Consider the population model

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{100}\right)\left(\frac{P}{25} - 1\right),$$

where $P(t)$ is the population at time t.

(a) For what values of P is the population in equilibrium?

$\frac{dP}{dt} = 0$ for $P = 0$, $P = 100$, $P = 25$

(b) For what values of P is the population increasing?

$\frac{dP}{dt} > 0$ for $25 < P < 100$

(c) For what values of P is the population decreasing?

$\frac{dP}{dt} < 0$ for $0 < 25 < P$

and $P > 100$.

2. Find all solutions of the differential equation

$$\frac{dy}{dt} = 2(y^2 - 4)te^{t^2}.$$

First, note that $y_1(t) = 2$ and $y_2(t) = -2$ are equilibrium solutions.

$$\int \frac{1}{(y-2)(y+2)}\,dy = \int 2t\,e^{t^2}\,dt$$

using partial fractions

$$\int \left[\frac{1/4}{y-2} - \frac{1/4}{y+2}\right] dy = e^{t^2} + C_1$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = e^{t^2} + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4e^{t^2} + C_2 \Rightarrow \left|\frac{y-2}{y+2}\right| = 4ke^{t^2}$$

3. Solve the initial-value problem

$$\frac{dy}{dx} = \frac{x-y}{x+y}, \quad y(1) = 1.$$

$$\frac{dy}{dx} = \frac{1 - y/x}{1 + y/x}.$$

let $v = y/x$. Then $y = xv$.

So $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - \frac{v(1+v)}{1+v}$$

$$= \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow \int \frac{1+v}{1-2v-v^2}\, dv = \int \frac{1}{x}\, dx$$

$u = 1-2v-v^2$

$du = -2-2v \; dv$

$= -2(1+v)\; dv \qquad \Rightarrow -\frac{1}{2}du = (1+v)\,dv$

3., cont.

$$\Rightarrow \int -\frac{1}{2} \cdot \frac{1}{u}\, du = \ln|x| + C$$

$$-\frac{1}{2} \ln|1-2v-v^2| = \ln|x| + C$$

$$-\frac{1}{2} \ln\left|1-\frac{2y}{x}-\left(\frac{y}{x}\right)^2\right| = \ln|x| + C$$

$$y(1)=1 \Rightarrow -\frac{1}{2} \ln|1-2-1| = \ln|1| + C$$

$$\Rightarrow -\frac{1}{2} \ln|-2| = C \Rightarrow C = -\frac{1}{2} \ln 2$$

$$\Rightarrow -\frac{1}{2} \ln\left|1-\frac{2y}{x}-\left(\frac{y}{x}\right)^2\right| = \ln|x| - \frac{1}{2} \ln 2.$$

This can be simplified to

$$y = -x + \sqrt{2x^2+2}$$