Math 333
Homework 2 Solutions

Numerical Simulations Problem. See posted Maple file.

1. Using Euler's method with $\delta t = 0.5$, $y(1) \approx 2.0625$.

2. Using Euler's method with $\delta t = 0.25$, $y(1) \approx 2.1079$.

3. Using Euler's method with $\delta t = 0.1$, $y(1) \approx 2.13859$.

4. Using the Runge-Kutta Fehlberg method in Maple, $y(1) \approx 2.16025$.

5. To find the exact value of $y(1)$, we solve the initial-value problem using the method of separation of variables. We obtain

$$y(t) = \sqrt{\frac{2}{3}t^3 + 4}.$$

Thus

$$y(1) = \sqrt{\frac{14}{3}} \approx 2.1602.$$

6. The Runge-Kutta Fehlberg method (the built-in numeric method in Maple) is the most accurate. The accuracy of Euler's method improves as the step size decreases. You might experiment with Maple to determine how many steps are needed so that Euler's method is as accurate as the RKF method.

Section 1.5

1.5 #4: Since $y_1(0) < y(0) < y_2(0)$, the solution $y(t)$ must satisfy $y_1(t) < y(t) < y_2(t)$ for all t, by the Uniqueness Theorem. Thus

$$-1 < y(t) < 1 + t^2$$

for all t.

1.5 #8: Note that $y(0) < 0$. Since $y_1(t) = 0$ is an equilibrium solution, the Uniqueness Theorem implies that $y(t) < 0$ for all t. Also, $dy/dt < 0$ for $y < 0$, so $y(t)$ is decreasing for all t. Thus $y(t) \to -\infty$ as $t \to \infty$ and $y(t) \to 0$ as $t \to -\infty$.

1.5 #13: The important observation for this problem is that the differential equation is not defined when $t = 0$.

(a) Note that $\frac{dy_1}{dt} = 0$ and $\frac{y_1}{t^2} = 0$, so $y_1(t)$ is a solution.

(b) Separating variables, we have

$$\int \frac{1}{y}\,dy = \int \frac{1}{t^2}\,dt.$$

Solving for y, we obtain

$$y(t) = ce^{-1/t},$$

where c is any constant. Thus, for any real number c, define the function $y_c(t)$ by

$$y_c(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ ce^{-1/t} & \text{for } t > 0 \end{cases}.$$

Then for each c, $y_c(t)$ satisfies the differential equation for all $t \neq 0$.

(c) Note that $f(t, y) = \frac{y}{t^2}$ is not defined at $t = 0$. Thus, we **cannot** apply the Uniqueness Theorem for the initial condition $y(0) = 0$.

1.5 #14: (a) The equation is separable. Solving for y, we obtain

$$y(t) = \frac{1}{\sqrt{c - 2t}},$$

where c is any constant. With the initial condition $y(0) = 1$, we obtain $c = 1$. Thus the solution of the IVP is

$$y(t) = \frac{1}{\sqrt{1 - 2t}}.$$

(b) This solution is defined for $t < 1/2$.

(c) As $t \to \frac{1}{2}^-$, the denominator of $y(t)$ becomes a small positive number, so $y(t) \to \infty$. As $t \to \infty$, $y(t) \to 0$.

Section 1.6

1.6 #2: The equilibrium points are $y = 0$ and $y = 1$. $y = 0$ is a sink and $y = 1$ is a source.

1.6 #8: The equilibrium points are $w = 0$ and $w = 4$. $w = 0$ is a node and $w = 4$ is a source.

1.6 #12: The equilibrium points are $w = \pm 1$ and $w = 0$. $w = -1$ is a source, $w = 0$ is a sink, and $w = 1$ is a source.

1.6 #14: Graph.

1.6 #20: Graph.

1.6 #37: (a) This phase line has three equilibrium points, $y = 0$, $y = -1$, and $y = 1$. $dy/dt < 0$ for $0 < y < 1$. Only equation (vii) satisfies these properties.

(b) This phase line has two equilibrium points, $y = 0$ and $y = 1$. $dy/dt \geq 0$ for $y > 0$ and $dy/dt < 0$ for $y < 0$. Only equation (ii) satisfies these properties.

(c) This phase line has two equilibrium points, $y = 0$ and $y = 2$. $dy/dt > 0$ for $0 < y < 2$. Only equation (vi) satisfies this property.

(d) This phase line has two equilibrium points, $y = 0$ and $y = 1$. $dy/dt < 0$ for $0 < y < 1$. Only equation (iii) satisfies this property.

1.6 #39: (a) There are three equilibrium points, $P = 0$, $P = 10$, and $P = 50$. A decreasing population at $P = 100$ implies that $f(P) < 0$ for $P > 50$. An increasing population at $P = 25$ implies that $f(P) > 0$ for $10 < P < 50$. Thus there are two possible phase lines since the arrow between $P = 0$ and $P = 10$ is undetermined.

(b) There are two basic types of graphs that go with the assumptions (though there are, of course, many specific examples). Both graphs should cross the P-axis at $P = 50$. One graph should also cross the axis at $P = 10$, and one should be tangent to the axis at $P = 10$.

(c) The functions $f(P) = P(P - 10)(50 - P)$ and $f(P) = P(P - 10)^2(50 - P)$ are two examples, but there are many others.

1.6 #41: The equilibrium points occur at solutions of

$$\frac{dy}{dt} = y^2 + a = 0.$$

There are three cases that we must consider. For $a > 0$, there are no equilibrium points. For $a = 0$, there is one equilibrium point, $y = 0$. For $a < 0$, there are two equilibrium points, $y = \pm\sqrt{a}$. To draw the phase lines, note that:

- In the case $a > 0$, $dy/dt = y^2 + a > 0$ for all y, so the solutions are *always increasing*.

- In the case $a = 0$, $dy/dt > 0$ for $y < 0$ and $dy/dt > 0$ for $y > 0$. Thus, $y = 0$ is a node.

- In the case $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$. Thus, $\sqrt{-a}$ is a source and $-\sqrt{-a}$ is a sink.

(a) The phase lines for $a < 0$ are qualitatively the same, and the phase lines for $a > 0$ are qualitatively the same.

(b) The phase line undergoes a qualitative change at $a = 0$.