# Math 333
# Homework 1 Solutions

## Section 1.1

**1.1 #4:** (a) The equilibrium solutions correspond to the values of $P$ for which $dP/dt = 0$ for all $t$. For the given equation, $dP/dt = 0$ if $P = 0$, $P = 50$, or $P = 200$.

(b) The population is increasing if $dP/dt > 0$. This occurs if $P < 0$ or $50 < P < 200$. Note that $P < 0$ should be considered as non-realistic for a population model.

(c) The population is decreasing if $dP/dt < 0$. This occurs if $0 < P < 50$ or $P > 200$.

**1.1 #8:** (a) Let $L_1(t)$ be the solution of the model with $L_1(0) = 1/2$ (the student who starts out knowing half of the list) and let $L_2(t)$ be the solution of the model with $L_2(0) = 0$ (the student who starts out knowing none of the list. At time $t = 0$,

$$\begin{aligned} \frac{dL_1}{dt} &= 1 \\ \frac{dL_2}{dt} &= 2. \end{aligned}$$

Thus the student who starts out knowing none of the list learns faster at time $t = 0$.

(b) The solution $L_2(t)$ with $L_2(0) = 0$ will learn one half of the list in some amount of time $t^\star > 0$. For $t > t^\star$, $L_2(t)$ will increase at exactly the same rate that $L_1(t)$ increases for $t > 0$. Thus, $L_2(t)$ increases at the same rate as $L_1(t)$ at $t^\star$ time units later. Hence, $L_2(t)$ will never catch up to $L_1(t)$, although they both approach 1 as t increases. Thus, $L_2(t) < L_1(t)$.

**1.1 #15:** Let $P(t)$ be the population at time t, $k$ the growth-rate parameter, and $N$ the carrying capacity. The modified models are:

(a) $dP/dt = kP(1 - P/N) - 100$

(b) $dP/dt = kP(1 - P/N) - P/3$

(c) $dP/dt = kP(1 - P/N) - a\sqrt{P}$, where $a$ is a positive parameter

**1.1 #16:** (a) The differential equation is

$$\frac{dP}{dt} = 0.3P(1 - \frac{P}{2500}) - 100.$$

The equilibrium solutions are

$$P_1 \cong 396$$

and

$$P_2 \cong 2104.$$

If $P > P_2$, then $dP/dt < 0$, so $P(t)$ is decreasing. If $P_1 < P < P_2$, then $dP/dt > 0$, so $P(t)$ is increasing. Thus the solution that satisfies the initial condition $P(0) = 2500$ decreases towards the equilibrium $P_2 \cong 2104$.

(b) The differential equation is

$$\frac{dP}{dt} = 0.3P(1 - \frac{P}{2500}) - \frac{P}{3}.$$

The equilibrium solutions are

$$P_1 \cong -277$$

and

$$P_2 = 0.$$

If $P > P_2 = 0$, then $dP/dt < 0$, so $P(t)$ is decreasing. Thus the solution that satisfies the initial condition $P(0) = 2500$ decreases towards the equilibrium $P = 0$ (extinction).

**1.1 #19:** (a) The term controlling the effect of the interaction of $x$ and $y$ on the rate of change of $x$ is $+\beta xy$. Since this term is *positive*, the presence of $y$'s helps the $x$ population grow. Thus, $x$ is the predator. Similarly, the term $-\delta xy$ in the $dy/dt$ equation implies that $y$ is the prey. If $y = 0$, then $dx/dt < 0$, so the predators will die out. Thus, the predators do not have sufficient alternate food sources. If $x = 0$, then $dy/dt > 0$ and the prey population increases exponentially.

(b) Since $-\beta xy$ is negative and $+\delta xy$ is positive, $x$ suffers due to its interaction with $y$ and $y$ benefits from its interaction with $x$. Thus, $x$ is the prey and $y$ is the predator. The predator does have other sources of food than the prey since $dy/dt = 0$ even if $x = 0$. Also, the prey has a limit on its growth due to the $-\alpha x^2/N$ term.

**1.1 #20:** (a) We consider $dx/dt$ in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number $x$ of prey is equal for both systems, $dx/dt$ is larger for system (i). Thus, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what effect the predators (represented by the $y$ terms) have on $dx/dt$ in each system. Since the magnitude of the coefficient of the $xy$-term is larger in system (ii) than in system (i), $y$ has a greater effect on $dx/dt$ in system (ii). Thus the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what effect the prey (represented by the $x$ terms) have on $dy/dt$ in each system. Since $x$ and $y$ are both non-negative, it follows that

$$-2y + \frac{1}{2}xy < -2y + 6xy.$$

Thus, if the number of predators is equal for both systems, $dy/dt$ is smaller in system (i). Thus, more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

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**Section 1.2**

**1.2 #2:** The equation is satisfied if we let

$$g(y) = \frac{\ln y}{2}.$$

**1.2 #10:** Use separation of variables to show that the general solution is

$$x(t) = \tan(t + c).$$

**1.2 #12:** Use separation of variables to show that the general solution is

$$y(t) = \pm(\frac{t^2}{3} + c)^{3/2}.$$

Note that this form does not include the equilibrium solution $y = 0$.

**1.2 #15:** First, note that the equilibrium solutions are $y = 0$ and $y = 1$. Next, suppose $y \neq 0$ and $y \neq 1$. Then we use separation of variables to obtain

$$\int \frac{1}{y(1-y)}\, dy = \int dt = t + c.$$

To integrate the right-hand side, we use partial fractions:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

We obtain $A = 1$ and $B = 1$. Thus,

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Thus we obtain

$$\int \frac{1}{y(1-y)}\, dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|.$$

Then

$$\ln\left|\frac{y}{1-y}\right| = t + c.$$

We solve for $y$ to obtain

$$y(t) = \frac{ke^t}{1+ke^t}.$$

**1.2 #18:** We use separation of variables to obtain

$$\int (y+1)\,dy = \int \frac{1}{t+1}\,dt.$$

Thus

$$\frac{y^2}{2} + y = \ln|t+1| + k.$$

We can solve using the quadratic formula (though it's not necessary) to obtain

$$y(t) = -1 \pm \sqrt{1 + 2\ln|t+1| + 2k} = -1 \pm \sqrt{2\ln|t+1| + c}.$$

**1.2 #26:** The solution of the initial value problem is

$$y(t) = -\frac{1}{\sqrt{1-2t^3/3}}.$$

**1.2 #30:** The solution of the initial value problem is

$$y(t) = -\sqrt{1+3e^{-2t}}.$$

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**Section 1.3**

**1.3 #2,3,9:** Slope fields in Maple.

**1.3 #16:**
(a) Since the slope field is constant on vertical lines, the given information is enough to draw the entire slope field.
(b) The solution with initial condition $y(0) = 2$ is a vertical translation of the given solution. We only need to change the constant of integration so that $y(0) = 2$.

**1.3 #18:**
(a) Since $y(t) = 2$ for all $t$ is a solution and $dy/dt = 0$ for all $t$, $f(t, y(t)) = f(t, 2) = 0$ for all $t$.
(b) Thus, the slope marks all have zero slope along the horizontal line $y = 2$.
(c) The graph of a solution must stay on the same side of the line $y = 2$ as it is at time $t = 0$.

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**Additional Problem.** Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}.$$

**Solution.** First note that the DE can be rewritten as

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Thus we make the change of variables $v = \frac{y}{x}$, or $y = xv$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx},$$

so the DE becomes

$$v + x\frac{dv}{dx} = 1 + v + v^2.$$

This DE is separable:

$$\frac{dv}{1+v^2} = \frac{dx}{x}.$$

Integrating, we obtain

$$\arctan v = \ln|x| + C.$$

Substituting for $v$, we obtain

$$\arctan(y/x) = \ln|x| + C.$$