Math 333

Homework 11 Solutions

1. $y'' - 2xy' + 2py = 0$.

Recall from class that the general recurrence relation is

$$a_{n+2} = \frac{-2(p-n)a_n}{(n+1)(n+2)}$$

(i) $p = 3 \Rightarrow y(0) = a_0 = 0, \ y'(0) = a_1 = 1$

$a_0 = 0 \Rightarrow a_{2n} = 0$ for $n \geq 0$.

$$a_1 = 1 \Rightarrow a_3 = \frac{-2(3-1)\cdot a_1}{2\cdot 3} = \frac{-2\cdot 2\cdot 1}{6} = -\frac{2}{3}$$

$$a_5 = \frac{-2(3-3)\cdot a_3}{4\cdot 5} = 0 \Rightarrow a_7 = a_9 = \cdots = 0$$

$a_{2n+1} = 0$ for $n \geq 2$.

$$\boxed{H_3(x) = x - \frac{2}{3}x^3}$$

(ii) $p=4 \Rightarrow y(0)=a_0=1$, $y'(0)=a_1=0$

$a_1=0 \Rightarrow a_3=a_5=\cdots=0$ $a_{2n+1}=0$ for $n\geq 0$.

$a_2=-pa_0=-4\cdot 1=-4$

$$a_4=\frac{-2(4-2)\cdot a_2}{3\cdot 4}=\frac{-2\cdot 2\cdot -4}{12}=\frac{16}{12}=\frac{8}{3}$$

$$a_6=\frac{-2(4-4)a_4}{5\cdot 6}=0 \Rightarrow a_8=a_{10}=\cdots=0$$ $a_{2n}=0$ for $n\geq 3$

$$\boxed{H_4(x)=1-4x^2+\frac{8}{3}x^4}$$

(iii) $p=5 \Rightarrow a_0=0$, $a_1=1$ $a_{2n}=0$ for $n\geq 0$.

$$a_3=\frac{-2(5-1)a_1}{2\cdot 3}=\frac{-2\cdot 4\cdot 1}{2\cdot 3}=-\frac{4}{3}$$

$$a_5=\frac{-2(5-3)a_3}{4\cdot 5}=\frac{-2\cdot 2\cdot -4}{4\cdot 5\cdot 3}=\frac{4}{15}$$

$a_7 = -\frac{2(5-5)a_5}{6 \cdot 7} = 0 \Rightarrow a_{2n+1} = 0$ for $n \geq 3$.

$$\boxed{H_5(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5}$$

2. $(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$

(a) $y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n \quad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n$

$$(1-x^2)\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \nu(\nu+1)\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2}$$

$$-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}\nu(\nu+1)a_n x^n = 0.$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} + \nu(\nu+1)a_n\right]x^n - \sum_{n=2}^{\infty}(n-1)n\,a_n x^n$$

$$-\sum_{n=1}^{\infty}2n\,a_n x^n = 0.$$

$$(2a_2 + v(v+1)a_0) + (6a_3 + v(v+1)a_1)x$$

$$+ \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} + v(v+1)a_n] x^n$$

$$- \sum_{n=2}^{\infty} (n-1)\cdot n \cdot a_n x^n - 2a_1 x - \sum_{n=2}^{\infty} 2na_n x^n = 0$$

$$2a_2 + v(v+1)a_0 = 0 \Rightarrow \boxed{a_2 = -\frac{v(v+1)a_0}{2}}$$

$$6a_3 + v(v+1)a_1 - 2a_1 = 0 \Rightarrow \boxed{a_3 = \frac{(-v(v+1)+2)a_1}{6}}$$

The general recurrence relation is:

$$(n+1)(n+2)a_{n+2} + v(v+1)a_n - (n-1)n\,a_n - 2na_n = 0$$

$$\boxed{a_{n+2} = \frac{(2n + n(n-1) - v(v+1))a_n}{(n+1)(n+2)}}$$

$$a_4 = \frac{(2\cdot 2 + 2\cdot 1 - v(v+1))a_2}{3\cdot 4} \Rightarrow \boxed{a_4 = \frac{(6 - v(v+1))a_2}{12}}$$

$$\Rightarrow \boxed{a_4 = \frac{(6-v(v+1))\cdot -v(v+1)a_0}{24}}$$

(b) $\nu = 0$: $a_0 = 1$, $a_1 = 0$

$a_1 = 0 \Rightarrow a_{2n+1} = 0$ for $n \geq 0$

$a_2 = 0 \Rightarrow a_{2n} = 0$ for $n \geq 1$.

$$\Rightarrow \boxed{P_0(x) = 1}$$

$$\boxed{P_1(x) = x}$$

$$\boxed{P_2(x) = 1 - 3x^2}$$

(c) $\nu = 3$: $a_0 = 0 \Rightarrow a_{2n} = 0$ for $n \geq 0$

$a_1 = 1$ $\quad a_3 = \frac{(2\cdot 1 + 1(1-1) - 3\cdot 4)a_1}{2\cdot 3} = \frac{-10a_1}{6} = -\frac{5}{3}a_1$

$a_5 = \frac{(2\cdot 3 + 3(2) - 3\cdot 4)a_3}{4\cdot 5} = 0 \Rightarrow a_{2n+1} = 0$ for $n \geq 2$

$$\boxed{P_3(x) = x - \frac{5}{3}x^3}$$

$\underline{\nu=4}$: $a_0 = 1$, $a_{2n+1} = 0$ for $n \geq 0$.

$$a_2 = \frac{(2\cdot 0 + 0\cdot -1 - 4\cdot 5)\,a_0}{1\cdot 2} = -10$$

$$a_4 = \frac{(2\cdot 2 + 2(2-1) - 4\cdot 5)\,a_2}{3\cdot 4} = \frac{-14\cdot -10}{12} = \frac{35}{3}$$

$$a_6 = \frac{(2\cdot 4 + 4\cdot 3 - 4\cdot 5)\,a_4}{5\cdot 6} = 0 \Rightarrow a_{2n} = 0 \text{ for } n \geq 3$$

$$\boxed{P_4(x) = 1 - 10x^2 + \frac{35}{3}x^4}$$

$\underline{\nu=5}$: $a_1 = 1$, $a_{2n} = 0$ for $n \geq 0$.

$$a_3 = \frac{(2\cdot 1 + 1\cdot 0 - 30)\,a_1}{2\cdot 3} = \frac{-28}{6} = -\frac{14}{3}$$

$$a_5 = \frac{(2\cdot 3 + 3\cdot 2 - 30)\,a_3}{4\cdot 5} = \frac{-18\cdot -14}{60} = \frac{21}{5}$$

$a_7 = 0 \Rightarrow a_{2n+1} = 0$ for $n \geq 3$.

$$\Rightarrow \boxed{P_5(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5}$$

$\underline{\nu=6}$: $a_0=1$, $a_{2n+1}=0$ for $n \geq 0$.

$$a_2 = \frac{(2\cdot 0 + 0\cdot -1 - 42)}{1\cdot 2} a_0 = -21$$

$$a_4 = \frac{(2\cdot 2 + 2\cdot 1 - 42)}{3\cdot 4} \cdot -21 = 63$$

$$a_6 = \frac{(2\cdot 4 + 4\cdot 3 - 42)}{5\cdot 6} \cdot 63 = -\frac{231}{5}$$

$$\Rightarrow \boxed{P_6(x) = 1 - 21x^2 + 63x^4 - \frac{231}{5}x^6}$$