Math 333
Homework 11: Series Solutions of Differential Equations, Part 1

1. Recall that the p-th Hermite polynomial $H_p(x)$ is found by solving the Hermite equation
$$y'' - 2xy' + 2py = 0, \tag{1}$$
with initial conditions given by
   - $y(0) = 1$ and $y'(0) = 0$ for even integers p
   - $y(0) = 0$ and $y'(0) = 1$ for odd integers p

   In class, we have shown that the first three Hermite polynomials are $H_0(x) = 1$, $H_1(x) = x$, and $H_2(x) = 1 - 2x^2$. Find the Hermite polynomials $H_3(x)$, $H_4(x)$, and $H_5(x)$ by solving the Hermite equation (1) for $p = 3$, $p = 4$, and $p = 5$ with initial conditions $y(0)$ and $y'(0)$ as specified above. Verify that the polynomials that you obtain indeed satisfy the differential equation. See Appendix B, p. 739 in your textbook for further discussion of solving Hermite's equation using series techniques.

2. (Problem 15, Appendix B). **Legendre's equation** is the second-order differential equation
$$(1 - x^2)y'' - 2xy' + \nu(\nu + 1)y = 0, \tag{2}$$
where ν is a constant. Note that $x = 0$ is an ordinary point of Eqn. (2), so that we can look for a power series solution centered at $x = 0$:
$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$
   (a) Compute the coefficients a_2, a_3, and a_4 in terms of a_0 and a_1. Try to find the general recurrence relation for the coefficients a_n. Hint: you should be able to express a_{n+2} in terms of a_n.

   (b) The Legendre polynomials are polynomial solutions $P_\nu(x)$ of Legendre's equation for positive integer values of ν corresponding to the following initial conditions:
   - $y(0) = 1$ and $y'(0) = 0$ for even integers ν
   - $y(0) = 0$ and $y'(0) = 1$ for odd integers ν.

   Show that the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = 1 - 3x^2$. Verify that these polynomials indeed satisfy the given differential equation and initial conditions.

   (c) Compute $P_\nu(x)$ for $\nu = 3,\ 4,\ 5,\ 6$. Verify that these polynomials indeed satisfy the given differential equation.