Math 333
Exam 2

- The exam is due at 11:30 am on Monday, May 5. NO LATE EXAMS WILL BE ACCEPTED.
- During the exam, you are permitted to use your textbook, your class notes, Maple (for simulation only), and any lecture notes or other reference material that I have posted on the course website. You may NOT use any other sources during the exam or discuss the exam with any other students or professors.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit.

Name:

"On my honor, I have neither given nor received any unauthorized aid on this examination. I understand that violation of this policy is against Kenyon College policy and will be prosecuted as an academic infraction."

Signature:

Question	Score	Maximum	Question	Score	Maximum
1		10	5		15
2		10	6		10
3		15	7		20
4		10	8		10
Bonus		10	**Total**		100

1. (10 points) Find the general solution of the differential equation

$$x^2y'' - 5xy' + 9y = 0.$$

2. (10 points) Find the general solution of the differential equation

$$(x-2)^2y'' + 5(x-2)y' + 8y = 0.$$

3. **Transformation of an Euler Equation to a Constant Coefficient Equation**. The Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

can be reduced to an equation with constant coefficients by a change of the independent variable. Let $x = e^z$, and consider the interval $x > 0$.

(a) (5 points) Show that the Euler equation becomes

$$\frac{d^2y}{dz^2} + (\alpha - 1)\frac{dy}{dz} + \beta y = 0.$$

(b) (5 points) The differential equation in part (a) is a second-order linear differential equation with constant coefficients. Find the general solution $y(z)$ of the differential equation in part (a). You will need to consider separately the cases in which the roots r_1 and r_2 of the characteristic polynomial are real and distinct, real and equal, and complex conjugates.

(c) (5 points) Finally, use the substitution $x = e^z$ to find the general solution $y(x)$ of the Euler equation. You can check that you obtain the same general solution that we obtained in class. Again, make sure to consider separately the different cases for r_1 and r_2.

4. (10 points) Find the general solution of the differential equation

$$x^2y'' - 2xy' + 2y = 3x^2 + 2\ln x, \quad x > 0.$$

5. Consider the differential equation

$$(x-1)y'' - xy' + y = 0, \quad x > 1.$$

(a) (5 points) Show that $y_1(x) = e^x$ is a solution of the differential equation.

(b) (10 points) Find the general solution of the differential equation. Depending on how you choose to solve this problem, you may find it useful to use Maple to evaluate integrals that arise in your computation. The syntax for evaluating the indefinite integral $\int f(x)\,dx$ in Maple is int(f(x),x).

6. (10 points) Find the general solution of the differential equation

$$y^{(4)} - 4y''' + 4y'' = e^{2t} + \cos(t).$$

7. **The Chebyshev Equation.** The Chebyshev differential equation is the second-order differential equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant. Note that $x = 0$ is an ordinary point of the differential equation, so we can look for a power series solution centered at $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

   (a) (10 points) Compute the coefficients a_2, a_3, and a_4 in terms of the arbitrary constants a_0 and a_1. Find the general recurrence relation for the coefficients a_n.

   (b) (10 points) The Chebyshev polynomials are polynomial solutions $P_\alpha(x)$ of the Chebyshev equation for positive integer values of α corresponding to the following initial conditions:

   - $y(0) = 1$ and $y'(0) = 0$ for even integers α.
   - $y(0) = 0$ and $y'(0) = 1$ for odd integers α.

   Find the Chebyshev polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$. Note that you can check your work by verifying that each polynomial indeed satisfies the differential equation.

8. (10 points) Consider the Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0,$$

where α and β are real constants. Find conditions on α and β so that:

   (a) All solutions approach zero as $x \to 0$.
   (b) All solutions are bounded as $x \to 0$.
   (c) All solutions approach zero as $x \to \infty$.
   (d) All solutions are bounded as $x \to \infty$.
   (e) All solutions are bounded both as $x \to 0$ and as $x \to \infty$.

Bonus: Some results on Legendre polynomials. (10 points)

Recall (from HW 11) that the Legendre polynomials are polynomial solutions $P_\nu(x)$ of Legendre's equation

$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$

for positive integer values of ν corresponding to the following initial conditions:

- $y(0) = 1$ and $y'(0) = 0$ for even integers ν
- $y(0) = 0$ and $y'(0) = 1$ for odd integers ν.

Since Legendre's equation is a second-order, linear, homogeneous differential equation, $kP_\nu(x)$ is also a solution for any constant k. For this problem, we'll normalize the Legendre polynomials $P_\nu(x)$ derived in HW 11 so that $P_\nu(1) = 1$ (note that this just means that we'll multiply each polynomial by an appropriate constant so that $P_\nu(1) = 1$).

(a) It can be shown that the general formula for $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k},$$

where $\lfloor n/2 \rfloor$ denotes the greatest integer less than or equal to $n/2$ (you do not need to derive this formula). By observing the form of $P_n(x)$ for n even and n odd, show that

$$P_n(-1) = (-1)^n.$$

(b) Show that the Legendre equation can also be written as

$$[(1-x^2)y']' = -\nu(\nu+1)y.$$

Then it follows that

$$[(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \text{ and } [(1-x^2)P_m'(x)]' = -m(m+1)P_m(x).$$

(c) By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \text{ if } n \neq m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the integral is $2/(2n+1)$.

(d) Given *any polynomial* f of degree n, it is possible to express f as a linear combination of $P_0, P_1, P_2, \ldots, P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of the part (c), show that

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x)P_k(x)\,dx.$$